AN

INTRODUCTORY TREATISE

ON THE

LUNAR THEORY

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LUNAR THEORY

BY

ERNEST W. BROWN, M.A.

PROFESSOR OF APPLIED MATHEMATICS IN HAVERFORD COLLEGE PA. U.S.A.
SOMETIME FELLOW OF CHRIST'S COLLEGE CAMBRIDGE

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PREFACE.

THE researches made during the last twenty years into the particular case of the Problem of Three Bodies, known as the Lunar Theory, have had the effect of creating a wider interest in a subject which had been somewhat neglected by the majority of mathematicians. Enquiry has been made, not only into the value of the various methods from a practical point of view, but also into questions which have an equal theoretical importance but which, until just lately, have been almost entirely neglected. The existence of integrals and of periodic solutions, and the representation of the solutions by infinite series, may be cited as instances.

In order to understand the bearing of these investigations on the lunar theory, some acquaintance with the older methods is desirable, if not necessary. In the following pages, an attempt has been made to supply a want in this direction, by giving the general principles underlying the methods of treatment, together with an account of the manner in which they have been applied in the theories of Laplace, de Pontécoulant, Hansen, Delaunay, and in the new method with rectangular coordinates. The explanation of these methods, and not the actual results obtained from them, having been my chief aim, only those portions of the developments and expansions, required for the fulfilment of this object, have been given.

The use of infinite series requires that investigations into their convergency should take an important place in any treatise on the Lunar Theory, and it is with regret that I have been obliged to leave it almost entirely aside, owing to the lack of any certain knowledge on the subject. The applications of the results to the formation of tables have shown that the series are of practical use, but the right to represent the solutions by

means of them has been discussed only for a few of the simpler cases, and the radius of convergence, when the series are arranged according to powers of any parameter, has been determined for elliptic motion only. It has also been found necessary to omit many other theoretical investigations which, in a more extended treatise, might have been included; but it is hoped that the references will cause the volume to be of service to those who desire to proceed to the study of these higher branches, as well to those who wish merely to obtain information concerning the older methods.

The difficulties of the subject are, perhaps, less inherent to it than due to the manner in which it has been presented to a student approaching it for the first time. The classical treatises are, almost invariably, original memoirs, and, as such, do not contain the details which are essential for a clear understanding of the scope and limitations of the problem in the form in which it is usually considered. Moreover, the authors generally confine themselves to their own methods, and the discovery of the relations which exist between the various forms of expression for the same function, is often troublesome. I have therefore given special attention to this point and also to another, closely associated with it, namely, the definitions and significations of the constants in disturbed motion.

A selection of one of the five methods, mentioned above, being necessary as a basis for the elucidation of the properties common to all, I have had no hesitation in adopting that of de Pontécoulant. Laplace takes the true longitude, instead of the time, as the independent variable—a method which renders the interpretation of the results difficult, until the reversion of series has been made—while the theories of Hansen and Delaunay are not well adapted to the end in view. De Pontécoulant's method of approximation being similar to that of Laplace, it then seemed to be sufficient, for the explanation of Laplace's treatment, to give the equations of motion, the first approximation, and a brief account of the manner in which the second and higher approximations are obtained.

In the Chapters on the methods of de Pontécoulant, Hansen and Delaunay, I have made some alterations in the form of presentation and the methods of proof, whenever these seemed to tend towards greater simplicity; where the differences from the original memoirs are important, they are noted. In order to facilitate references to the latter, the original notations are adhered to as far as possible, and this has necessitated the

employment of three distinct notations. The tables given at the end of the volume will show the Chapters in which they are severally employed, and will enable the reader to find, without difficulty, the meaning of any frequently recurring symbol.

The first four Chapters respectively contain investigations of the form of the disturbing function, of the equations of motion, of the expansions relating to elliptic motion, and of the methods adopted in order to approximate to the solution. The term 'intermediary' is used to signify any orbit which may be adopted as a first approximation to the path actually described—a definition somewhat different from that given by Prof. Gyldèn. In Chap. V. the equations for the variations of the elements in disturbed motion are obtained in an elementary way and also by the more elegant and symmetrical method of Jacobi. The properties and methods of development of the disturbing function are collected in Chap. VI.

The details of the second and of parts of the third approximation to the solution of de Pontécoulant's equations will be found in Chap. VII.; the inequalities are divided into classes in order to show their origin more clearly. Chap. VIII., devoted to the arbitrary constants, is made intentionally simple and explicit.

Chaps. IX. and X. contain the theories of Delaunay and Hansen, respectively. A special effort has been made to free the methods of the latter from the difficulties and obscurities which surround them in the *Fundamenta* and the *Darlegung*. In Chap. XI. I have attempted to give a complete method for the treatment of the solar inequalities in the Moon's motion, based on that initiated by Dr Hill for those parts of them which depend only on the ratios of the mean solar and lunar motions. The infinite determinant, which arises in the calculation of the principal parts of the mean motions of the perigee and the node, is considered at some length, the conditions for its convergency and its development in series being included. Chap. XII. contains an account, necessarily brief, of methods other than those previously discussed.

In Chap. XIII. the inequalities arising from planetary action and from the ellipticity of the Earth are considered. It being impossible to give an adequate account of these in the space at my disposal, they are treated by Dr Hill's modification of Delaunay's method only. An exception is made in favour of the inequalities due to the motion of the ecliptic, and of

the inequality known as the secular acceleration, since the effects of these appeared to be more simply explained by other methods.

The various memoirs and treatises of which I have made use are referred to in the text. In particular, the excellent collection of methods, contained in the first and third volumes of the *Mécanique Celeste* of M. F. Tisserand, has been frequently consulted.

I take this opportunity of acknowledging a deep debt of gratitude to Professor G. H. Darwin and Dr E. W. Hobson, not only for their valuable criticisms and suggestions made while reading the proof-sheets of this work, but also for their advice and assistance rendered freely at all times during the last eight years. My thanks are also due to Mr P. H. Cowell, Fellow of Trinity College, Cambridge, for much help in the correction of all the proof-sheets and in the verification of the formulæ and results.

I may add that the cooperation of the officers of the University Press has made it possible for me to see the printing almost completed during my temporary residence in Cambridge, and thus to avoid the delays and difficulties which would otherwise have arisen.

ERNEST W. BROWN.

HAVERFORD COLLEGE,

1895, *December* 13.

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ERRATA.

Page 5.	Head-line.	*For* MOON'S		*read* SUN'S.	
„ 60.	Line 1.	„	a	„	a^2.
„ 108.	„ 35.	„	$+\frac{33}{16}m^4-$	„	$+\frac{33}{16}m^4+$.
„ 109.	„ 8.	„	$-\frac{33}{16}m^4+$	„	$-\frac{33}{16}m^4-$.
„	„ 10.	„	$\frac{19}{256}m^4$	„	$\frac{25}{256}m^4$.

CHAPTER I.

FORCE-FUNCTIONS.

1. THE Newtonian law of attraction states that either of two particles exerts on the other a force which is proportional directly to the product of their masses and inversely to the square of the distance between them. Let $\mathfrak{F}$ be the force between two particles of masses m, m' placed at a distance r from one another, then

$$\mathfrak{F} = k\frac{mm'}{r^2},$$

where k is a constant depending only on the units employed. It is known as the Gaussian constant of attraction.

If we use any terrestrial unit of mass, k will vary directly as the unit of mass and the square of the unit of time and inversely as the cube of the unit of length.

The accelerative effect of the force exerted by m on m' is

$$\mathfrak{A} = k\frac{m}{r^2}.$$

The *Astronomical unit of mass* corresponding to given units of length and time, is obtained by so choosing this unit that $k = 1$. Since $\mathfrak{A}$ is an acceleration, when the units of mass, length and time vary, the astronomical unit of mass varies directly as the cube of the unit of length and inversely as the square of the unit of time. It is used very largely in theoretical investigations in astronomy and the frequent repetition of the constant k is thereby avoided.

For further remarks on this unit see E. J. Routh, *Analytical Statics*, Vol. II. pp. 1, 2, 3.

For the sake of brevity the term 'force' is often used to indicate 'accelerative effect of a force.' In general, no confusion will arise in this use of the word. With an astronomical unit of mass, an acceleration may appear either as a mass divided by the square of a length or, in its usual form, as a length divided by the square of a time.

2. The general problem of Celestial Mechanics consists in the determination of the relative motions of p bodies attracting one another according to the Newtonian law. This problem is not able to be solved directly: in order to deal with it, certain limitations must be made.

The first simplification which we shall introduce, is made by eliminating from consideration the effects of the size, shape and internal distribution of mass of the bodies. A well-known theorem in attractions states, 'that a spherical body, of which the density is the same at the same distance from its centre, attracts a similarly constituted spherical body as if the mass of each were condensed into a particle situated at its centre of figure.' The larger bodies of our solar system differ but little from spheres in shape and their centres of mass are certainly but little distant from their centres of figure—if at all; the shapes of very small bodies when under the attraction of a large one play little part in their motions. We assume therefore that the bodies are spherical and attract one another in the same way as particles of equal mass situated at their centres of figure. With these limitations the problem is known as that of p bodies.

Again, owing to the conditions under which the bodies of our solar system move, we are further able to divide the problem of p bodies into several others, each of which may be treated as a case of the problem of *three* particles, or, as it is generally called, the *Problem of Three Bodies.*

The greater part of the *Lunar Theory* is a particular case of the Problem of Three Bodies; it involves the determination of the motion of the Moon relative to the Earth, when the mutual attractions of the Earth, Moon and Sun, considered as particles, are the only forces under consideration. When this has been found, the effects produced by the actions of the planets, the non-spherical forms of the bodies etc., can be exhibited as small corrections to the coordinates.

3. We proceed to consider the impressed forces in the problem of three particles. There are two methods of treating them, from the combination of which a suitable form of force-function can be obtained. In the first method, we find the accelerations due to the forces acting on the Sun and Moon relatively to the Earth, and in the second method, those acting on the Moon relatively to the Earth and on the Sun relatively to the centre of mass of the Earth and Moon.

(i) *The forces relative to the Earth.*

In figure 1, let E, M, m', G respectively denote the places of the Earth, Moon, Sun and the centre of mass of the Earth and Moon.

Let the masses of the Earth, Moon and Sun measured in astronomical

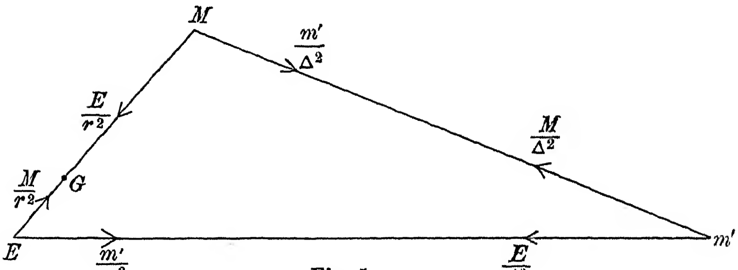

Fig. 1.

units be E, M, m'. Let the distances, ME, $m'E$, $m'M$ be r, r', Δ: these must always be considered as positive quantities.

We shall only deal here with the *accelerations due to the forces* and, in accordance with the remark made in Art. 1, speak of them as *forces*.

The forces acting on E are M/r^2 in the direction EM and m'/r'^2 in the direction Em'. Similar expressions hold for the forces acting on M and m', as shown in the figure. Hence the forces acting, relatively to the Earth, are,

On the Moon,				On the Sun,			
$\frac{E+M}{r^2}$	in the	direction of	ME,	$\frac{E+m'}{r'^2}$	in the	direction of	$m'E$,
$\frac{m'}{\Delta^2}$	,,	,,	Mm',	$\frac{M}{\Delta^2}$	,,	,,	$m'M$,
$\frac{m'}{r'^2}$	,,	,,	$m'E$;	$\frac{M}{r^2}$	,,	,,	ME.

Take the Earth as the origin of a set of rectangular axes. Let x, y, z, and x', y', z', be the coordinates of M and m', and $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$, the forces acting on M and m' relatively to E, referred to these axes. We have then,

$$\mathfrak{X} = -\frac{E+M}{r^2}\frac{x}{r} - \frac{m'}{\Delta^2}\frac{x-x'}{\Delta} - \frac{m'}{r'^2}\frac{x'}{r'},$$

$$\mathfrak{X}' = -\frac{E+m'}{r'^2}\frac{x'}{r'} - \frac{M}{\Delta^2}\frac{x'-x}{\Delta} - \frac{M}{r^2}\frac{x}{r},$$

with corresponding expressions for $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{Y}'$, $\mathfrak{Z}'$.

If we put

$$F = \frac{E+M}{r} + \frac{m'}{\Delta} - m'\frac{xx'+yy'+zz'}{r'^3},$$

$$F' = \frac{E+m'}{r'} + \frac{M}{\Delta} - M\frac{xx'+yy'+zz'}{r^3},$$

we shall have

$$\mathfrak{X} = \frac{\partial F}{\partial x},\quad \mathfrak{Y} = \frac{\partial F}{\partial y},\quad \mathfrak{Z} = \frac{\partial F}{\partial z};\qquad \mathfrak{X}' = \frac{\partial F'}{\partial x'},\quad \mathfrak{Y}' = \frac{\partial F'}{\partial y'},\quad \mathfrak{Z}' = \frac{\partial F'}{\partial z'}.$$

The expressions denoted by F, F' are the force-functions for the motions of the Moon and the Sun relative to the Earth.

Let the cosine of the angle MEm' be denoted by S. Then

$$xx' + yy' + zz' = rr'S,$$

and therefore F, F' are independent of the particular set of axes—fixed or moving—which may be used.

The expressions for F, F' are quite general, and when they shall have been substituted in the six differential equations which represent the motion of M, m' relative to E, we shall have the general equations of motion for the problem of three particles. (It has been tacitly assumed that the system is free from any external influence having a tendency to disturb the relative motion of the three particles.) It is not possible to integrate these equations except in special cases. In order to obtain F in the most suitable form, certain observations must be made before proceeding further.

4. In finding the motion of the Moon, the first assumption usually made consists in considering the motion of the Earth about the Sun or, what is the same thing, of the Sun about the Earth as previously known and therefore the coordinates x', y', z' as known functions of the time. Now the function F' which is used to find the motion of the Sun contains the unknown coordinates of the Moon. It becomes necessary to see what effect those terms in F' which have M as a factor produce in the relative motion of the Sun.

If we limit F' to its first term $(E+m')/r'$, the resulting motion of m' about E will be elliptic. Now in the lunar theory, r is very small compared to r' or Δ while r' and Δ are of nearly equal magnitude. Hence it appears that the third term in F' is that which will cause most deviation from elliptic motion. This term is due to the force M/r^2, which is one of the forces acting on E. If we had referred the motion of m' to G this term would not have entered. We shall therefore find the motion of m' relative to G and discuss the other terms in the new force-functions for m' and M.

(ii) *The forces acting on m' relatively to G and on M relatively to E.*

5. We have
$$EG = \frac{M}{E+M} EM,$$

and therefore the accelerations of G relative to E are $M/(E+M)$ times those of M relative to E. Hence the force on m' relative to G, parallel to the axis of x, is $\mathfrak{X}' - \mathfrak{X}M/(E+M)$ which, by Art. 3, is equal to

$$-\frac{E+M+m'}{E+M}\left(\frac{E}{r'^2}\frac{x'}{r'} + \frac{M}{\Delta^2}\frac{x'-x}{\Delta}\right).$$

Let x_1, y_1, z_1, be the coordinates of m' referred to parallel axes through G: let $m'G = r_1$. Then

$$x_1 = x' - \frac{M}{E+M}x, \quad y_1 = y' - \frac{M}{E+M}y, \quad z_1 = z' - \frac{M}{E+M}z$$

and therefore
$$x' - x = x_1 - \frac{E}{E+M}x, \text{ etc.}$$

Also

$$r'^2 = \left(x_1 + \frac{M}{E+M}x\right)^2 + \left(y_1 + \frac{M}{E+M}y\right)^2 + \left(z_1 + \frac{M}{E+M}z\right)^2,$$

$$\Delta^2 = \left(x_1 - \frac{E}{E+M}x\right)^2 + \left(y_1 - \frac{E}{E+M}y\right)^2 + \left(z_1 - \frac{E}{E+M}z\right)^2.$$

If therefore we put

$$F_1' = \frac{E+M+m'}{E+M}\left(\frac{E}{r'} + \frac{M}{\Delta}\right),$$

where r', Δ are now expressed in terms of x, y, z, x_1, y_1, z_1, the differentials $\frac{\partial F_1'}{\partial x_1}, \frac{\partial F_1'}{\partial y_1}, \frac{\partial F_1'}{\partial z_1}$ will be the forces acting on m' relatively to G.

Again replacing in Art. 3, x', y', z' by their values above, we have

$$\mathfrak{X} = -\frac{E+M}{r^2}\frac{x}{r} + \frac{m'}{\Delta^3}\left(x_1 - \frac{E}{E+M}x\right) - \frac{m'}{r'^3}\left(x_1 + \frac{M}{E+M}x\right).$$

The forces $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ can therefore be derived, by partial differentiation with respect to x, y, z, from the force-function

$$F_1 = \frac{E+M}{r} + \frac{m'}{\Delta}\frac{E+M}{E} + \frac{m'}{r'}\frac{E+M}{M},$$

where Δ, r' are expressed as before in terms of x, y, z, x_1, y_1, z_1.

It is not difficult to show by expanding F_1' in powers of r/r_1, that it differs little from $(E+M+m')/r_1$. We have, putting $\cos MGm' = S_1$,

$$r'^2 = r_1^2 + 2\frac{M}{E+M}rr_1S_1 + \left(\frac{M}{E+M}\right)^2 r^2,$$

whence
$$\frac{1}{r'} = \frac{1}{r_1} - \frac{M}{E+M}\frac{r}{r_1^2}S_1 + \left(\frac{M}{E+M}\right)^2 \frac{r^2}{r_1^3}(\tfrac{3}{2}S_1^2 - \tfrac{1}{2}) + \ldots.$$

Similarly

$$\frac{1}{\Delta} = \frac{1}{r_1} + \frac{E}{E+M}\frac{r}{r_1^2}S_1 + \left(\frac{E}{E+M}\right)^2 \frac{r^2}{r_1^3}(\tfrac{3}{2}S_1^2 - \tfrac{1}{2}) + \ldots$$

and therefore

$$F_1' = \frac{m' + E + M}{r_1}\left[1 + \frac{E\,.\,M}{(E+M)^2}\frac{r^2}{r_1^2}(\tfrac{3}{2}S_1^2 - \tfrac{1}{2}) + \dots\right].$$

Now r/r_1 differs little from $\frac{1}{400}$ at any time and M/E is approximately $\frac{1}{80}$. Hence the order of the second term of F_1' relatively to the first is roughly

$$\frac{1}{80}\,\frac{1}{400^2} = \frac{1}{12{,}800{,}000},$$

a quantity which may be neglected here. We can then neglect the second and higher terms of F_1' and consequently assume that *the motion of m' about G is an ellipse.*

6. The Moon thus produces little effect on the motion of the centre of mass of the Earth and Moon and consequently we can consider this point as moving in an ellipse in accordance with Kepler's laws. The actions of the planets, however, produce effects which become very marked after long periods of time. These effects, being exhibited by terms in the expressions of x_1, y_1, z_1, are transmitted to the Moon through F. We ought properly to have considered the problem of p bodies or even less generally, to have included in the force-functions F, F' the forces produced by all the planets. But the action, both direct and indirect, of the other planets on the Moon is so small that in most cases it may be neglected: where it should be considered, it is always possible to do so in the form of small corrections. The variations produced in the motion of the Earth or of G by the other planets belong properly to the planetary theory and need not be considered here. All we require here concerning the motion of the Earth is, that it should be supposed *known*. The only reason for considering its force-function at all, is to see how the unknown coordinates of the Moon enter therein and to eliminate them as far as possible.

It is a remarkable fact, and one which shows the extraordinary care required in the treatment of the lunar and the planetary theories, that the direct actions of the planets on the Moon are in general much less marked than their indirect actions as transmitted through the Earth. These indirect effects, though sometimes too small to be detected in the motion of the Earth, may become sufficiently large to be observed in the motion of the Moon.

7. With the assumption that the motion of G is elliptic, we ought properly to use the force-function F_1 for finding the motion of the Moon. But as it is generally found more convenient to use F, we shall expand both functions in order to see in what way the latter must be corrected to give the former.

The expansions of $1/r'$ and $1/\Delta$ given in Art. 5, furnish

$$\frac{1}{E\Delta} + \frac{1}{Mr'} = \frac{1}{r_1}\left[\frac{1}{E} + \frac{1}{M} + \frac{1}{(E+M)}\frac{r^2}{r_1^2}(\tfrac{3}{2}S_1^2 - \tfrac{1}{2}) + \frac{E-M}{(E+M)^2}\frac{r^3}{r_1^3}(\tfrac{5}{2}S_1^3 - \tfrac{3}{2}S_1) + \dots\right].$$

When this is substituted in F_1, the term

$$\frac{1}{r_1}\left(\frac{1}{E}+\frac{1}{M}\right)m'(E+M)$$

will contribute nothing to the forces since it is independent of the coordinates of the Moon: it may therefore be omitted. We shall then have

$$F_1=\frac{E+M}{r}+\frac{m'}{r_1}\left[\frac{r^2}{r_1^2}\left(\tfrac{3}{2}S_1^2-\tfrac{1}{2}\right)+\frac{E-M}{E+M}\frac{r^3}{r_1^3}\left(\tfrac{5}{2}S_1^3-\tfrac{3}{2}S_1\right)+\dots\right].$$

Again, we find by expanding F in powers of r/r' and omitting the useless term m'/r',

$$F=\frac{E+M}{r}+\frac{m'}{r'}\left[\frac{r^2}{r'^2}\left(\tfrac{3}{2}S^2-\tfrac{1}{2}\right)+\frac{r^3}{r'^3}\left(\tfrac{5}{2}S^3-\tfrac{3}{2}S\right)+\dots\right].$$

Now x', y', z', r' refer to the motion of the Sun about the Earth, and x_1, y_1, z_1, r_1 to that about G, and these are contained in F, F_1 respectively in the same way. If therefore in F we consider x', y', z', r' to refer to the motion about G, which motion is supposed known in terms of the time, F, F_1 will be the same except as regards the ratio M/E.

Let now a, a' be the mean values of r, r_1 (or r'). It will be found later that expansions will be made in powers of a/a' and that the parts resulting from the term in F,

$$\frac{m'}{r'}\frac{r^2}{r'^2}\left(\tfrac{3}{2}S^2-\tfrac{1}{2}\right),$$

do not contain a/a'. The parts resulting from the term in F,

$$\frac{m'}{r'}\frac{r^3}{r'^3}\left(\tfrac{5}{2}S^3-\tfrac{3}{2}S\right),$$

contain a/a' in its first power. Comparing F, F_1 we see that if in the results produced by using F_1, in addition to the change noted above, we replace the ratio a/a' wherever it occurs by

$$\frac{E-M}{E+M}\frac{a}{a'},$$

we shall be able to use F instead of F_1. The terms containing $(a/a')^2$ we should multiply by $(E-M)^2/(E+M)^2$, and so on.

This does not quite account for all the differences between F, F_1. The terms of next highest order in F, F_1 are respectively

$$+\frac{m'}{r'}\frac{r^4}{r'^4}\left(\tfrac{35}{8}S^4-\tfrac{15}{4}S^2+\tfrac{3}{8}\right)$$

and

$$+\frac{E^2-EM+M^2}{(E+M)^2}\frac{m'}{r_1}\frac{r^4}{r_1^4}\left(\tfrac{35}{8}S_1^4-\tfrac{15}{4}S_1^2+\tfrac{3}{8}\right).$$

The first expression must be multiplied by

$$(E-M)^2/(E+M)^2.$$

Hence, after the changes previously noted, there will still remain to be added to F, the term

$$\frac{EM}{(E+M)^2}\frac{m'}{r'}\frac{r^4}{r'^4}\left(\tfrac{35}{8}S^4-\tfrac{15}{4}S^2+\tfrac{3}{8}\right).$$

Now the order of this remainder, compared to the first of those terms in F which depend on the action of the Sun, is

$$\frac{M}{E}\frac{a^2}{a'^2}=\frac{1}{12,800,000}$$

approximately. The largest periodic term produced by the Sun has a coefficient in longitude of less than 5000″, so that it is very improbable that such a small quantity can produce any appreciable effect. The effect of the differences in the higher terms will be still smaller. We may therefore conclude that the replacing of a/a' by $a(E-M)/a'(E+M)$ will sufficiently account for the remaining differences.

8. The problem of the Moon's motion is therefore reduced to the determination of the motion of a particle of mass M, under the action of a true force-function MF, where

$$F=\frac{E+M}{r}+m'\left(\frac{1}{\Delta}-\frac{xx'+yy'+zz'}{r'^3}\right),$$

in which x', y', z' are the known functions of the time obtained from purely elliptic motion. We shall now consider E instead of G as the origin.

Let $E+M=\mu$, and put $\qquad F=\dfrac{\mu}{r}+R.$

The quantity R depends on the action of the Sun and is known as the *Disturbing Function*.

There is now no further need to consider the functions F_1, F', F_1'.

9. The distinction between the planetary and the lunar theories is one of analysis only, based on certain facts deduced from observations on the nature of the motion of the bodies forming the solar system. Both theories are particular cases of the problem of three bodies which, owing to the deficiencies of our methods of analysis, is at present only capable of being solved by tedious expansions, even when the bodies are so favourably placed relatively to one another as those which come within the range of observation. Almost nothing is at present known of the possible curves which bodies of any masses and placed at any distances from one another may describe. In the case of the planetary theory, where it is required to investigate the perturbations of one planet by another, or more properly, the mutual perturbations of two planets, we can use the functions F_1, F_1'

where E stands for the mass of the Sun and M, m' for the masses of the planets. For in this case the ratios $(E-M)/(E+M)$, $(E-m')/(E+m')$ are so near unity owing to the great mass of the Sun compared to that of any of the planets and the actual perturbations are so small, that the differences of these ratios from unity can in general be neglected.

Rough observations extending over a sufficient interval of time show very quickly that, during that interval at least, the planets describe curves which approximate more or less closely to circles of which the Sun occupies the centres. A more exact representation of their motion is given by Kepler's well-known laws. The known satellites, and in particular the Moon, also approximately satisfy these laws with reference to the planets about which they move, but for a shorter time; they also exhibit larger deviations from them. Observation too has shown that the eccentricities of their ellipses and the mutual inclinations of the planes of motion of all the principal planets oscillate about mean values which are in no case very great. The same is true of the Moon and of most of the satellites with reference to the orbits of their primaries.

It is assumed then that expansions may be made in powers of the eccentricities and of suitable functions of the inclinations. But when we are considering the perturbations of one body produced by another, it has just been seen that expansion will also be made in powers of the ratio of the distances of the disturbed and disturbing body from the primary.

It is at this point that the first separation of the lunar and the planetary theories takes place. In the lunar theory, the distance from the primary—the Earth—of the disturbing body—the Sun—is very great compared to that of the disturbed body—the Moon, and we naturally expand first in powers of this ratio in order that we may start with as few terms as possible. In the planetary theory, the distances of the disturbed and the disturbing bodies—two planets—from the Sun which is the primary, may be a large fraction. For example the mean distances of Venus and the Earth from the Sun are approximately in the ratio 7 : 10, and in order to secure sufficient accuracy when expanding in powers of this ratio, a very large number of terms would have to be taken. In the case of the planetary theory then, we delay expansion in powers of the ratio of the distances as long as possible and form the series first in powers of the eccentricities and inclinations.

Again, in the lunar theory the mass of the disturbing body is very great compared to that of the primary, a ratio on which it is evident the magnitude of the perturbations greatly depends. On the other hand, in the planetary theory the disturbing body has a very small mass compared to that of the primary. From these facts we are led to expect that large terms will be present in the expressions for the motion of the Moon due to the action of the Sun and, from the remarks made above, that the later terms in the expansions will decrease rapidly; and in the planetary theory we expect large numbers of terms of nearly the same magnitude, none of them being very great. This expectation however is largely modified by some further remarks about to be made.

In the integrations performed in both theories the coefficients of the periodic terms by means of which the coordinates are expressed, become frequently much greater than might have been expected *a priori*. In the lunar theory, before this can happen in such a way as to cause much trouble, the coefficients have previously become so small that it is not necessary to consider such terms beyond a certain limit. Suppose in the planetary theory that n, n' are the mean motions of the two planets round the primary. Then coefficients will, for example, continually be having multipliers of the form $n'/(in \pm i'n')$ and $n'^2/(in \pm i'n')^2$ produced by integration (i, i' positive integers). In general, the greater i, i' are, the smaller will be the corresponding coefficient. But owing to the two facts that the ratio

$a : a'$ may be nearly unity and that the ratio $n : n'$ may very nearly approach that of two small whole numbers, a coefficient may become very great. For example, five revolutions of Jupiter are very nearly equal in time to two of Saturn, while the ratio of their mean distances is roughly 6 : 11. One result is a periodic inequality which has a coefficient of 28′ in the motion of Jupiter and 48′ in that of Saturn. Such inequalities take a long period to run through all their values, the one in question having a periodic time equal to about 76 revolutions of Jupiter or 913 years, so that the variation due to this term in one revolution is small. The periods of the principal terms in the motion of the Moon are generally short but some of them have large coefficients, so that the deviations from elliptic motion are well marked.

One of the greatest difficulties in the planetary theory, perhaps owing chiefly to our methods of expression, is the presence, in the values of the coordinates (when the latter are obtained as functions of the time), of terms which increase continually with the time, and thus, after the lapse of a certain interval, render the expressions for the coordinates useless as a representation of the motion. Whether such terms can be eliminated by the use of suitable functions is not at present certain. Recently the work of Gyldén* has gone far in the direction of achieving this object. In the lunar theory, the difficulty also occurs, but, as regards the perturbations produced by the Sun, is easily bridged by means of a slight artifice.

It will readily be conceived, from the few statements made here, that in general, different methods will be required for treating the two problems of a satellite disturbed by the Sun and of a planet disturbed by another planet. When the disturbing planet is an inferior one, we use a function corresponding to F, but we have then to develope in powers of a'/a instead of a/a'. In the cases of some minor planets again special methods are required owing to the great eccentricity of their orbits. All the problems are essentially the same: the analytical difficulties alone compel us to treat them differently.

As concerns questions of purely mathematical interest, the planetary theory has in the past opened out a larger field for the investigator than the lunar theory. In the last few years however the researches of Hill†, Adams†, Poincaré‡ and others have brought the latter problem forward again and given it a new stimulus.

10. Let $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$ be the coordinates, referred to any origin and any rectangular axes, of three bodies of masses m_1, m_2, m_3 which attract one another according to the Newtonian law. Let r_{12}, r_{23}, r_{13} be the distances between them. The force acting on m_1, resolved parallel to the axis of x, is

$$\frac{m_1 m_2}{r_{12}^2}\frac{x_2 - x_1}{r_{12}} + \frac{m_1 m_3}{r_{13}^2}\frac{x_3 - x_1}{r_{13}};$$

the forces acting on m_1 are therefore derivable from the function

$$\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}},$$

* *Acta Mathematica*, Vols. VII., XI., XV. etc., also *Traité analytique des Orbites absolues, etc.* Stockholm, 1893.

† See Chapter XI.

‡ *Les Méthodes nouvelles de la Mécanique Céleste*, Paris, Vol. I., 1892, Vol. II., 1893. These researches are outside the scope of this book.

or, since r_{23} does not contain the coordinates x_1, y_1, z_1, from the function

$$F = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}.$$

The symmetry of this expression shows that it is also the force-function for the motions of m_2 and m_3.

Generally, if there be p bodies, it is easily seen that

$$F = \Sigma_i \Sigma_j \frac{m_i m_j}{r_{ij}},$$

where i, j receive values 1, 2... p, the terms for which $i = j$ being excluded.

CHAPTER II.

THE EQUATIONS OF MOTION.

11. THE methods used in the solution of the lunar problem may be roughly divided into four classes. In the first class we may place those methods in which the time is taken as the independent variable, the radius vector (or its inverse), the true longitude measured on a fixed plane, and the tangent of the latitude above this plane, as dependent variables; the equations of motion are expressed in terms of these quantities and solved by continued approximation with elliptic motion as a basis, so as to exhibit these coordinates as functions of the time and the arbitrary constants introduced by integration. Under this heading we may include the theories of Lubbock and de Pontécoulant. In the methods of the second class four similar variables are used, but the true longitude is taken to be the independent variable and the other three variables are expressed in terms of it. A reversion of series is finally necessary in order to obtain the coordinates in terms of the time. Clairaut, d'Alembert, Laplace, Damoiseau and Plana followed this plan.

A third class will embrace those methods in which the Moon is supposed to be moving at any time in an ellipse of variable size, shape and position; this is known as the method of the Variation of Arbitrary Constants and it was used in different ways by Euler in his first theory, by Poisson and with great success by Hansen and Delaunay. In the fourth class may be placed those theories in which rectangular coordinates referred to moving axes are used, with the time as independent variable. Euler's second theory and the general methods resulting from the works of Hill and Adams may be included under this heading.

We shall give here the equations of motion used by de Pontécoulant and Laplace and the generalised form of Hill's equations, to illustrate the methods of the first, second and fourth classes respectively: the principles which form the basis of the methods of the third class will be given in Chapter V. We shall also include here some considerations on the general problem of three bodies with special reference to the known integrals.

The methods used by de Pontécoulant, Delaunay and Hansen will be found in Chapters VII., IX. and X. respectively; the methods of Hill and Adams with the extensions to the complete problem are given in Chapter XI. A short summary of the methods employed by other lunar theorists is made in Chapter XII.

(i) *De Pontécoulant's equations of motion.*

12. Let x, y, z be the coordinates of the Moon referred to three rectangular axes, fixed in direction and passing through the centre of the Earth. The equations of motion of the Moon will be, according to Newton's laws of motion, since $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ are, by Art. 8, the forces parallel to the axes,

$$M\ddot{x} = M\frac{\partial F}{\partial x}, \qquad M\ddot{y} = \frac{\partial F}{\partial y}, \qquad M\ddot{z} = \frac{\partial F}{\partial z},$$

or by the same article,

$$\ddot{x} + \frac{\mu x}{r^3} = \frac{\partial R}{\partial x}, \qquad \ddot{y} + \frac{\mu y}{r^3} = \frac{\partial R}{\partial y}, \qquad \ddot{z} + \frac{\mu z}{r^3} = \frac{\partial R}{\partial z},$$

where R is supposed expressed in terms of x, y, z, x', y', z'.

Let the plane of (xy) be the plane of the Sun's motion, supposed fixed, and let the axis of x be a fixed line in this plane. Let x, y, z, M, M' be the

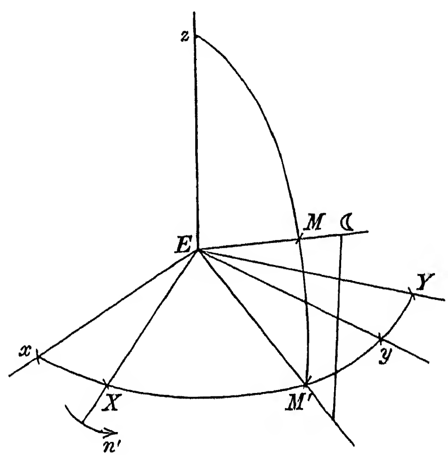

Fig. 2.

points where a sphere of unit radius cuts the axes, the radius vector and its projection on the (xy) plane, respectively. Let

r be the radius vector of the Moon,

r_1 its projection,

v the longitude of this projection reckoned from the axis of x, and

s the tangent of the latitude of the Moon above the plane of (xy).

Hence $v = xM'$, $s = \tan M'M$,

and therefore

$$x = \frac{r\cos v}{\sqrt{1+s^2}}, \qquad y = \frac{r\sin v}{\sqrt{1+s^2}}, \qquad z = \frac{rs}{\sqrt{1+s^2}} = r_1 s, \qquad r_1 = \frac{r}{\sqrt{1+s^2}}.$$

If we change the variables from x, y, z to r_1, v, z, the equations of motion become

$$\left.\begin{aligned} \ddot{r}_1 - r_1\dot{v}^2 &= -\frac{\mu r_1}{r^3} + \frac{\partial R}{\partial r_1}, \\ \frac{1}{r_1}\frac{d}{dt}(r_1^2\dot{v}) &= \frac{1}{r_1}\left(\frac{\partial R}{\partial v}\right), \\ \ddot{z} &= -\frac{\mu z}{r^3} + \frac{\partial R}{\partial z}, \end{aligned}\right\} \quad \text{.......................... (1)},$$

where $\left(\frac{\partial R}{\partial v}\right)$ denotes partial differentiation of R when R is expressed as a function of r_1, v, z, in contradistinction to $\frac{\partial R}{\partial v}$ which will denote partial differentiation when R is expressed as a function of r, v, s; the two expressions are easily seen to be equal though differently expressed.

Multiplying these equations by $2\dot{r}_1$, $2r_1\dot{v}$, $2\dot{z}$, respectively, and adding, we obtain

$$\frac{d}{dt}\left(\dot{r}_1^2 + r_1^2\dot{v}^2 + \dot{z}^2 - \frac{2\mu}{r}\right) = 2\frac{d'R}{dt},$$

where $$d'R = \frac{\partial R}{\partial r_1}dr_1 + \frac{\partial R}{r_1\partial v}r_1\,dv + \frac{\partial R}{\partial z}dz.$$

Whence, if a be a constant, we have on integration

$$\dot{r}_1^2 + r_1^2\dot{v}^2 + \dot{z}^2 = \frac{2\mu}{r} - \frac{\mu}{a} + 2\int\frac{d'R}{dt}dt \quad \text{.................. (2)},$$

the expression for the square of the velocity.

The integral on the right-hand side of this equation requires explanation. From the way in which $d'R$ was formed, it is evident that $d'R/dt$ denotes differentiation of R with respect to t, only in so far as t is present through the variability of the coordinates r_1, v, z and not through its presence explicitly in the coordinates of the Sun. We suppose then that $d'R/dt$ has been formed in this way and we can put

$$\int\frac{d'R}{dt}dt = \int d'R,$$

an equation which defines the meaning of $\int d'R$.

In order that the integration may be actually performed, $d'R/dt$ ought to be expressed in general as a function of the time only. It will, however, contain the unknown coordinates r_1, v, z, and its value can only be obtained by a process of gradual approximation to the values of these coordinates.

Multiplying the first of equations (1) by r_1, the third by z and adding the sum to equation (2), we obtain

$$r_1\ddot{r}_1 + \dot{r}_1^2 + z\ddot{z} + \dot{z}^2 = \frac{\mu}{r} - \frac{\mu}{a} + r_1 \frac{\partial R}{\partial r_1} + z\frac{\partial R}{\partial z} + 2\int d'R.$$

But since we have $r_1^2 + z^2 = r^2$ and therefore $r_1\ddot{r}_1 + \dot{r}_1^2 + z\ddot{z} + \dot{z}^2 = \frac{1}{2}d^2(r^2)/dt^2$, the equation becomes

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2) - \frac{\mu}{r} + \frac{\mu}{a} = r_1\frac{\partial R}{\partial r_1} + z\frac{\partial R}{\partial z} + 2\int d'R \quad\ldots\ldots\ldots\ldots\ldots (3).$$

Also from the second of equations (1), if h be a constant, we obtain

$$r_1^2\dot{v} = h + \int\left(\frac{\partial R}{\partial v}\right)dt.$$

whence

$$\dot{v} - \frac{h(1+s^2)}{r^2} = \frac{1+s^2}{r^2}\int\left(\frac{\partial R}{\partial v}\right)dt \quad\ldots\ldots\ldots\ldots\ldots (4).$$

Finally, from the third of equations (1) we have, substituting the value of z in terms of r, s,

$$\frac{d^2}{dt^2}\left(\frac{rs}{\sqrt{1+s^2}}\right) + \frac{\mu}{r^3}\frac{rs}{\sqrt{1+s^2}} = \frac{\partial R}{\partial z} \quad\ldots\ldots\ldots\ldots\ldots (5).$$

13. We must now express $\frac{\partial R}{\partial r_1}, \left(\frac{\partial R}{\partial v}\right), \frac{\partial R}{\partial z}$ in terms of $\frac{\partial R}{\partial r}, \frac{\partial R}{\partial v}, \frac{\partial R}{\partial s}$ where, in the former three R is considered a function of r_1, v, z, and in the latter three a function of r, v, s. Let δr, δv, δz be any independent variations of r_1, v, z and δr, δv, δs, δR the corresponding variations of r, v, s, R. We have

$$\frac{\partial R}{\partial r_1}\delta r_1 + \left(\frac{\partial R}{\partial v}\right)\delta v + \frac{\partial R}{\partial z}\delta z = \delta R = \frac{\partial R}{\partial r}\delta r + \frac{\partial R}{\partial v}\delta v + \frac{\partial R}{\partial s}\delta s.$$

Also, since $r^2 = r_1^2 + z^2$, $s = z/r_1$, and therefore

$$r\delta r = r_1\delta r_1 + z\delta z, \quad \delta s = \delta z/r_1 - z\delta r_1/r_1^2,$$

we obtain by substituting for δr, δs in the previous equation and equating the corresponding coefficients of the independent variations δr_1, δv, δz,

$$\frac{\partial R}{\partial r_1} = \frac{r_1}{r}\frac{\partial R}{\partial r} - \frac{z}{r_1^2}\frac{\partial R}{\partial s}, \quad \left(\frac{\partial R}{\partial v}\right) = \frac{\partial R}{\partial v}, \quad \frac{\partial R}{\partial z} = \frac{z}{r}\frac{\partial R}{\partial r} + \frac{1}{r_1}\frac{\partial R}{\partial s}.$$

These furnish immediately

$$r_1\frac{\partial R}{\partial r_1} + z\frac{\partial R}{\partial z} = r\frac{\partial R}{\partial r}, \quad \frac{\partial R}{\partial z} = \frac{s}{\sqrt{1+s^2}}\frac{\partial R}{\partial r} + \frac{\sqrt{1+s^2}}{r}\frac{\partial R}{\partial s}.$$

Substituting these results in equations (3), (4), (5), we obtain

$$\left.\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}(r^2)-\frac{\mu}{r}+\frac{\mu}{a}&=r\frac{\partial R}{\partial r}+2\int d'R\\ \dot{v}-\frac{h(1+s^2)}{r^2}&=\frac{1+s^2}{r^2}\int\frac{\partial R}{\partial v}\,dt\\ \frac{d^2}{dt^2}\left(\frac{rs}{\sqrt{1+s^2}}\right)+\frac{\mu}{r^3}\frac{rs}{\sqrt{1+s^2}}&=\frac{\sqrt{1+s^2}}{r}\frac{\partial R}{\partial s}+\frac{s}{\sqrt{1+s^2}}\frac{\partial R}{\partial r}\end{aligned}\right\}\ldots\ldots\text{(A)},$$

the equations of motion sought.

14. The first of these three equations contains no differentials, with respect to the time, of s, v, while the second has none of r, s: the third equation has differentials of both r, s. But since it is found by observation that the arbitrary constants introduced by integration are such that r differs from a constant by a small quantity only and that s is itself always a small quantity, we shall see that the three equations are respectively useful in determining the radius vector, the longitude of the projection of the radius vector on the fixed plane and the latitude above this plane. They will therefore be referred to as the radius-, longitude- and latitude-equations respectively. The equations (A) having been first used by de Pontécoulant* as a basis for the lunar theory, are referred to under his name. Equations of an almost identical form were, however, obtained by Laplace in Chap. VI., Book II. of the *Mécanique Céleste.*

15. One of the chief difficulties of the lunar theory is the interpretation of the arbitrary constants arising from the integration of the differential equations. It is necessary, in order that we may be able to find them accurately from observation, to have them exhibited in such a way that their physical signification can be *exactly* fixed. Special stress is laid on this point. The same care is necessary also when it is desired to compare the results of one mode of development with another, in order that the relations between the constants used in the two sets of results may be determined.

It will be noticed that the equations of motion have been reduced to two of the second order and one of the first; the integration of them will therefore give rise to five arbitrary constants. These five constants with the two h, a, already introduced, will make seven, while our original equations of motion—three of the second order—only demand six. There must therefore be some relation between these seven constants after the integrations have been performed and, to determine it, use must be made of some other combination of the original equations of motion. For this purpose we have

* *Système du Monde*, vol. IV. No. 1.

from the sum of the first and third of equations (1) multiplied respectively by r_1, z,

$$r_1\ddot{r}_1 + z\ddot{z} - r_1^2\dot{v}^2 = -\frac{\mu}{r} + r_1\frac{\partial R}{\partial r_1} + z\frac{\partial R}{\partial z}.$$

But since $r_1^2 + z^2 = r^2$, $s = \tan U$ (where, for a moment, U denotes the latitude) we obtain

$$\dot{r}_1^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{U}^2$$

and therefore

$$r_1\ddot{r}_1 + z\ddot{z} = r\ddot{r} + \dot{r}^2 - \dot{r}_1^2 - \dot{z}^2 = r\ddot{r} - r^2\dot{U}^2 = r\ddot{r} - r^2\dot{s}^2/(1+s^2)^2.$$

Also
$$r_1\frac{\partial R}{\partial r_1} + z\frac{\partial R}{\partial z} = r\frac{\partial R}{\partial r}.$$

Whence, after division by r^2 we obtain, since $r_1^2 = r^2/(1+s^2)$,

$$\frac{\ddot{r}}{r} - \frac{\dot{v}^2}{1+s^2} - \frac{\dot{s}^2}{(1+s^2)^2} + \frac{\mu}{r^3} = \frac{1}{r}\frac{\partial R}{\partial r} \quad\text{.....................} (6).$$

When the motion is elliptic, that is, when we neglect the right-hand sides of equations (A), the relation between the seven constants is very easily found (see Chapter III.). When the general equations of motion are treated, the equation (6) may be used to find the required relation. Six of the constants are determined in the disturbed motion so as to simplify the interpretation of the final results as much as possible (Chapter VIII.). The presence of a seventh constant greatly assists us in this respect.

(ii) *Laplace's Equations of Motion.*

16. Let $\mathfrak{P}_1$, $\mathfrak{T}_1$, $\mathfrak{Z}_1$ be the forces acting on the Moon, resolved parallel to the direction of the projection of the radius vector on the fixed plane, perpendicular to this direction in the fixed plane and perpendicular to the fixed plane respectively. The equations of motion will be

$$\ddot{r}_1 - r_1\dot{v}^2 = \mathfrak{P}_1,$$

$$\frac{1}{r_1}\frac{d}{dt}(r_1^2\dot{v}) = \mathfrak{T}_1,$$

$$\frac{d^2}{dt^2}(r_1 s) = \mathfrak{Z}_1.$$

Let $u_1 = 1/r_1$ and $r_1^2\dot{v} = H$. Transforming so as to make the independent variable v, we obtain* from the first two equations, since $dt = r_1^2 dv/H$,

$$\frac{d^2u_1}{dv^2} + u_1 = -\frac{\mathfrak{P}_1}{H^2u_1^2} - \frac{\mathfrak{T}_1}{H^2u_1^3}\frac{du_1}{dv} \quad\text{.....................} (7).$$

* Tait and Steele, *Dynamics of a Particle*, 4th Ed., Art. 136.

Also, since $\dot{H} = \mathfrak{T}_1 r_1$, we have

$$H\frac{dH}{dv} = H\frac{\dot{H}}{\dot{v}} = \frac{\mathfrak{T}_1}{u_1^3}.$$

Integrating, we obtain, if h be an arbitrary constant,

$$H^2 = h^2 + 2\int\frac{\mathfrak{T}_1}{u_1^3}\,dv\,;$$

whence

$$\frac{dt}{dv} = \frac{1}{Hu_1^2} = \frac{1}{hu_1^2}\left(1 + 2\int\frac{\mathfrak{T}_1}{h^2u_1^3}\,dv\right)^{-\frac{1}{2}} \quad\text{..................}\ (8),$$

and equation (7) may be written,

$$\frac{d^2u_1}{dv^2} + u_1 = -\frac{\mathfrak{P}_1}{h^2u_1^2} - \frac{\mathfrak{T}_1}{h^2u_1^3}\frac{du_1}{dv} - 2\left(\frac{d^2u_1}{dv^2} + u_1\right)\int\frac{\mathfrak{T}_1}{h^2u_1^3}\,dv \quad\text{......}\ (9).$$

Since $H = h$ when $\mathfrak{T}_1$ is zero, h will have the same meaning as in (i) when $R = 0$.

Again from the first and third equations of motion we obtain

$$-u_1(\mathfrak{P}_1 s - \mathfrak{Z}_1) = \ddot{s} + 2\dot{r}_1\dot{s}u_1 + s\dot{v}^2$$

$$= \frac{d^2s}{dv^2}\dot{v}^2 + \frac{ds}{dv}\ddot{v} + 2\frac{ds}{dv}\frac{\dot{v}\dot{r}_1}{r_1} + s\dot{v}^2$$

$$= \dot{v}^2\left(\frac{d^2s}{dv^2} + s\right) + \frac{ds}{dv}\left(\frac{\dot{H}}{r_1^2} - \frac{2H}{r_1^3}\dot{r}_1\right) + 2\frac{ds}{dv}\frac{H}{r_1^3}\dot{r}_1$$

$$= \dot{v}^2\left(\frac{d^2s}{dv^2} + s\right) + \mathfrak{T}_1 u_1\frac{ds}{dv},$$

where the values $\ddot{v} = \dfrac{d}{dt}\left(\dfrac{H}{r_1^2}\right)$, $\dot{H} = \mathfrak{T}_1 r_1$ have been substituted. Since

$$\dot{v}^2 = H^2u_1^4 = h^2u_1^4(1 + 2\int\mathfrak{T}_1\,dv/h^2u_1^3),$$

there results,

$$\frac{d^2s}{dv^2} + s = -\frac{\mathfrak{P}_1 s - \mathfrak{Z}_1}{h^2u_1^3} - \frac{\mathfrak{T}_1}{h^2u_1^3}\frac{ds}{dv} - 2\left(\frac{d^2s}{dv^2} + s\right)\int\frac{\mathfrak{T}_1}{h^2u_1^3}\,dv \quad\text{......}\ (10).$$

Also, since F (Art. 8) is a function of x, y, z and therefore of u_1, v, s, we have, by the principle of virtual displacements,

$$\mathfrak{P}_1\delta r_1 + \mathfrak{T}_1 r_1\delta v + \mathfrak{Z}_1\delta z = \delta F = \frac{\partial F}{\partial u_1}\delta u_1 + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial s}\delta s\,;$$

and since $\delta r_1 = -\delta u_1/u_1^2$, $\delta z = \delta s/u_1 - s\delta u_1/u_1^2$, we obtain by equating the coefficients of the independent displacements δu_1, δv, δs,

$$\frac{\partial F}{\partial u_1} = -\frac{\mathfrak{P}_1 + s\mathfrak{Z}_1}{u_1^2}, \qquad \frac{\partial F}{\partial v} = \frac{\mathfrak{T}_1}{u_1}, \qquad \frac{\partial F}{\partial s} = \frac{\mathfrak{Z}_1}{u_1}.$$

Substituting for $\mathfrak{P}_1$, $\mathfrak{T}_1$, $\mathfrak{Z}_1$ in equations (9), (10), (8), we obtain,

$$\left.\begin{aligned}
\frac{d^2u_1}{dv^2}+u_1&=\frac{1}{h^2}\frac{\partial F}{\partial u_1}+\frac{s}{h^2u_1}\frac{\partial F}{\partial s}-\frac{1}{h^2u_1^2}\frac{du_1}{dv}\frac{\partial F}{\partial v}-\frac{2}{h^2}\left(\frac{d^2u_1}{dv^2}+u_1\right)\int\frac{\partial F}{\partial v}\frac{dv}{u_1^2}\\
\frac{d^2s}{dv^2}+s&=\frac{s}{h^2u_1}\frac{\partial F}{\partial u_1}+\frac{1+s^2}{h^2u_1^2}\frac{\partial F}{\partial s}-\frac{1}{h^2u_1^2}\frac{ds}{dv}\frac{\partial F}{\partial v}-\frac{2}{h^2}\left(\frac{d^2s}{dv^2}+s\right)\int\frac{\partial F}{\partial v}\frac{dv}{u_1^2}\\
\frac{dt}{dv}&=\frac{1}{hu_1^2}\left(1+\frac{2}{h^2}\int\frac{\partial F}{\partial v}\frac{dv}{u_1^2}\right)^{-\frac{1}{2}}
\end{aligned}\right\}\text{(11)},$$

which are the equations found and used by Laplace*.

17. We have just obtained the equations of motion in the form of two equations of the second order and one of the first, furnishing five arbitrary constants on integration; these with the constant h will form the six constants necessary. The form of these equations renders them very useful in certain departments of the lunar theory. For a complete development of the perturbations produced by the Sun, with the accuracy demanded by observation to-day, they are, nevertheless, almost excluded by the fact that, after u_1, s, t have been found in terms of v, a reversion of series is necessary to get v—the most important coordinate—in terms of t. This last process would probably demand as much labour as that necessary to find the other coordinates and the time in terms of v.

(iii) *Equations of Motion referred to moving rectangular axes*†.

18. Take axes EX, EY moving in the fixed plane of (xy) with angular velocity n' round the axis of z (Fig. 2, p. 13). Let X, Y, z be the coordinates of the Moon and X', Y', 0 those of the Sun referred to these axes, so that, as before, we assume the fixed plane of (XY) or (xy) to be the ecliptic. The equations of motion of the Moon will be, according to the usual formulæ for accelerations referred to moving axes,

$$\left.\begin{aligned}
\ddot{X}-2n'\dot{Y}-n'^2X&=\frac{\partial F}{\partial X}\\
\ddot{Y}+2n'\dot{X}-n'^2Y&=\frac{\partial F}{\partial Y}\\
\ddot{z}&=\frac{\partial F}{\partial z}
\end{aligned}\right\}\text{..........................(12)}.$$

If we define the function F', by the equation,

$$F'=\tfrac{1}{2}n'^2(X^2+Y^2)+F \text{ .... (13)},$$

* *Mécanique Céleste*, Books II. 15; VII. 1.

† See remarks on Euler's second theory in Chapter XII.

the three equations of motion may be written,

$$\ddot{X} - 2n'\dot{Y} = \frac{\partial F'}{\partial X}, \qquad \ddot{Y} + 2n'\dot{X} = \frac{\partial F'}{\partial Y}, \qquad \ddot{z} = \frac{\partial F'}{\partial z}.$$

Let now $\quad v = X + Y\sqrt{-1}, \qquad \sigma = X - Y\sqrt{-1}.$

We have then

$$\frac{\partial F'}{\partial X} + \frac{\partial F'}{\partial Y}\sqrt{-1} = 2\frac{\partial F'}{\partial \sigma}, \qquad \frac{\partial F'}{\partial X} - \frac{\partial F'}{\partial Y}\sqrt{-1} = 2\frac{\partial F'}{\partial v}.$$

Multiply the second equation of motion by $\sqrt{-1}$: the first two equations of motion become, by addition and subtraction,

$$\ddot{v} + 2n'\dot{v}\sqrt{-1} = 2\frac{\partial F'}{\partial \sigma}, \qquad \ddot{\sigma} - 2n'\dot{\sigma}\sqrt{-1} = 2\frac{\partial F'}{\partial v}.$$

Again put $\quad \zeta = e^{(n-n')(t-t_0)\sqrt{-1}},$

and change the independent variable from t to ζ; n, t_0 are two constants at present not defined. Let

$$\nu = n - n', \qquad D = \zeta\frac{d}{d\zeta}, \qquad \mathrm{m} = \frac{n'}{n-n'} = \frac{n'}{\nu}.$$

We have

$$\frac{d}{dt} = \nu\sqrt{-1}\;\zeta\frac{d}{d\zeta} = \nu\sqrt{-1}\,D, \qquad \frac{d^2}{dt^2} = -\nu^2 D \,.\, D = -\nu^2 D^2,$$

according to the usual notation for operators. Substituting these values in the equations for v, σ, they become,

$$D^2 v + 2\mathrm{m}Dv = -\frac{2}{\nu^2}\frac{\partial F'}{\partial \sigma}, \qquad D^2\sigma - 2\mathrm{m}D\sigma = -\frac{2}{\nu^2}\frac{\partial F'}{\partial v} \quad \text{.....(14).}$$

19. We must now develope F'. From the expression of F given in Art. 7, we have, since $E + M = \mu$,

$$F = \frac{\mu}{r} + \frac{m'}{r'^3}\left(\tfrac{3}{2}r^2S^2 - \tfrac{1}{2}r^2\right) + \frac{m'}{r'^3}\frac{1}{r'}\left(\tfrac{5}{2}r^3S^3 - \tfrac{3}{2}rS\,.\,r^2\right) + \dots.$$

Let now n', $2a'$ be the mean motion and major axis of the ellipse described by the Sun. It will be shown in Chapter III. that we can put $m' = n'^2a'^3$. Let $\kappa = \mu/\nu^2$. With these substitutions, we obtain from equation (13), since $\mathrm{m} = n'/\nu$, $X^2 + Y^2 = v\sigma$, $r^2 = v\sigma + z^2$,

$$\left.\begin{aligned}\frac{2}{\nu^2}F' &= \frac{2\kappa}{r} + \mathrm{m}^2 v\sigma + \mathrm{m}^2\frac{a'^3}{r'^3}\left(3r^2S^2 - v\sigma - z^2\right) + \frac{\mathrm{m}^2}{a'}\frac{a'^4}{r'^4}\left\{5r^3S^3 - 3rS\left(v\sigma + z^2\right)\right\} + \dots \\ &= \frac{2\kappa}{r} + \tfrac{3}{4}\mathrm{m}^2(v+\sigma)^2 - \mathrm{m}^2 z^2 + \Omega\end{aligned}\right\}$$

$$\text{.....................(15),}$$

where

$$\Omega = 3\mathrm{m}^2 \left\{\frac{a'^3}{r'^3} r^2 S^2 - \tfrac{1}{4}(v+\sigma)^2\right\} - \mathrm{m}^2 (v\sigma + z^2)\left(\frac{a'^3}{r'^3} - 1\right) + \frac{\mathrm{m}^2}{a'}\frac{a'^4}{r'^4}\{5r^3S^3 - 3rS(v\sigma + z^2)\} + \dots$$
$$= \Omega_2 + \Omega_3 + \dots \qquad \text{.....................(16).}$$

In this last expression Ω_p stands for the terms of degree p in powers and products of v, σ, z, and therefore of degree $-p+2$ in powers of a' (r' is of the same degree as a').

Substituting the value of F' given by the second of equations (15) in equations (14), the latter become, since $r^2 = v\sigma + z^2$,

$$\left.\begin{aligned} D^2 v + 2\mathrm{m}Dv + \tfrac{3}{2}\mathrm{m}^2(v+\sigma) - \frac{\kappa v}{r^3} &= -\frac{\partial\Omega}{\partial\sigma} \\ D^2\sigma - 2\mathrm{m}D\sigma + \tfrac{3}{2}\mathrm{m}^2(v+\sigma) - \frac{\kappa\sigma}{r^3} &= -\frac{\partial\Omega}{\partial v} \end{aligned}\right\} \qquad \text{............(17),}$$

and the third equation of motion, after changing the independent variable, is

$$D^2 z - \mathrm{m}^2 z - \frac{\kappa z}{r^3} = -\frac{1}{2}\frac{\partial\Omega}{\partial z} \qquad \text{.......................(18).}$$

The equations (17), (18) form the basis for a general treatment of the lunar theory. We shall now give Hill's transformation in its most general form.

20. Multiply the first of equations (17) by σ, the second by v and subtract. We obtain

$$D(vD\sigma - \sigma Dv - 2\mathrm{m}v\sigma) + \tfrac{3}{2}\mathrm{m}^2(v^2 - \sigma^2) = \sigma\frac{\partial\Omega}{\partial\sigma} - v\frac{\partial\Omega}{\partial v} \qquad \text{......(19).}$$

Again, adding equations (17) with the same multipliers to equation (18) multiplied by $2z$, we have

$$\sigma D^2 v + vD^2\sigma + 2zD^2 z - 2\mathrm{m}(vD\sigma - \sigma Dv) + \tfrac{3}{2}\mathrm{m}^2(v+\sigma)^2 - 2\mathrm{m}^2 z^2 - \frac{2\kappa}{r}$$
$$= -\left(\sigma\frac{\partial\Omega}{\partial\sigma} + v\frac{\partial\Omega}{\partial v} + z\frac{\partial\Omega}{\partial z}\right) = -\sum_2^\infty p\Omega_p \qquad \text{......(20).}$$

The last result is arrived at by applying Euler's theorem for homogeneous functions to equation (16).

Further, multiplying equations (17) and (18) by $D\sigma$, Dv, $2Dz$ respectively, and adding, the result may be put into the form*

$$D\left[Dv\,.\,D\sigma + (Dz)^2 + \tfrac{3}{4}\mathrm{m}^2(v+\sigma)^2 - \mathrm{m}^2 z^2 + \frac{2\kappa}{r}\right] = -\left(\frac{\partial\Omega}{\partial\sigma}D\sigma + \frac{\partial\Omega}{\partial v}Dv + \frac{\partial\Omega}{\partial z}Dz\right)$$
$$\text{...............(21).}$$

* The term $Dv\,.\,D\sigma$ is the product of Dv and $D\sigma$ and must not be confounded with $D(vD\sigma)$. Similarly $(Dz)^2 = Dz \times Dz$.

But since Ω is expressible in terms of the coördinates of the Moon and the Sun and since those of the latter are supposed known functions of the time or of ζ, we may suppose Ω expressed as a function of v, σ, z, ζ. We therefore have

$$D\Omega = \frac{\partial \Omega}{\partial v} Dv + \frac{\partial \Omega}{\partial \sigma} D\sigma + \frac{\partial \Omega}{\partial z} Dz + \frac{\partial \Omega}{\partial \zeta} D\zeta.$$

But
$$\frac{\partial \Omega}{\partial \zeta} D\zeta = \zeta \frac{\partial \Omega}{\partial \zeta} = D_t \Omega = D(D^{-1} D_t \Omega),$$

where D_t denotes the operation D performed on Ω with reference to the portions which contain t (or ζ) explicitly, and D^{-1} is the inverse operation to D. Substituting these results in equation (21) and integrating, we have

$$Dv \,.\, D\sigma + (Dz)^2 + \tfrac{3}{4}\mathrm{m}^2 (v+\sigma)^2 - \mathrm{m}^2 z^2 + \frac{2\kappa}{r} = C - \Omega + D^{-1}(D_t \Omega) \ldots (22),$$

where C is a constant.

Adding this to equation (20), we obtain (since $\Omega = \Omega_2 + \Omega_3 + \ldots$) an equation which may be put into the form,

$$D^2(v\sigma + z^2) - Dv \,.\, D\sigma - (Dz)^2 - 2\mathrm{m}(vD\sigma - \sigma Dv) + \tfrac{9}{4}\mathrm{m}^2(v+\sigma)^2 - 3\mathrm{m}^2 z^2$$
$$= C - \sum_2^\infty (p+1)\Omega_p + D^{-1}(D_t\Omega) \ldots\ldots (23).$$

The three equations (18), (19), (23) are the generalised form of Hill's equations (see Chapter XI.).

21. It will be noticed that these three differential equations are each of the second order and therefore on integration will furnish six arbitrary constants. A constant of integration C has already been introduced, while κ or μ has disappeared from (19), (23),—the equations which furnish the motion in the fixed plane. There is therefore a relation containing κ, between these seven arbitrary constants. This relation will be determined from one of our original equations of motion. The constants n, t_0 introduced into the equations will be defined in Chapter XI. as two of the arbitraries of the solution.

The advantage possessed by the equations (19), (23), which are of principal importance for the determination of v, σ, arises chiefly from the fact that their left-hand members are homogeneous quadratic functions of v, σ, z. When we neglect the parallax of the Sun, that is, when we consider the Sun to be at an infinite distance, the right-hand members of the equations are also of the same form except as regards the constant C. Even when terms depending on the distance of the Sun are included, since it is not generally necessary to take them beyond the order $1/a'^2$, the terms thus added will only be of the third and fourth degrees in v, σ, z. Equation (18)

has not this form, but it is not difficult to obtain an equation of a form similar to (19), free from the divisor r^3.

The remarks of this last paragraph apply equally to equations (19), (23), when they are expressed in terms of the real variables X, Y, z, t. The use of the conjugate complexes υ, σ enables us however to put our solution in an algebraic form. It will be seen later that X, Y are expressible respectively by means of cosines and sines of the same multiples of t. As a consequence of this, υ, σ are expressible in series, with ζ as the variable and with real coefficients. Also, σ can be derived from υ by putting $1/\zeta$ for ζ, so that it is only necessary to calculate either υ or σ. The advantage of algebraic over trigonometric series, when the multiplication of two series is in question, will be easily understood.

Since $$D\upsilon\,.\,D\sigma+(Dz)^2=-(\dot{X}^2+\dot{Y}^2+\dot{z}^2)/\nu^2,$$
the equation (22) is the Jacobian integral referred to moving axes. When the solar eccentricity is neglected the term $D^{-1}\,(D_t\Omega)$ vanishes. We may therefore look upon this term as the variation of the constant of Energy due to the eccentricity of the Sun's orbit.

Also since $$\upsilon\sigma+z^2=r^2,\quad \upsilon D\sigma-\sigma D\upsilon=-(\dot{Y}X-\dot{X}Y)/\nu,$$
we can express immediately equations (19) and (23) in a real form.

Although, either of the equations (17) is, since υ, σ are complex quantities, a complete substitute for the first two of equations (12) the same cannot be said of equations (19), (23). The reason of this is easily seen. If we give to υ, σ, ζ their values in terms of X, Y, t, each of the equations (17) furnishes a real and an imaginary part. On the other hand when the same substitutions are made in (19), (23), the former gives an imaginary part only and the latter a real part only.

22. There are two particular cases of equations (12) which require notice and, in order to treat them, we must know something further about the disturbing function.

We have from Art. 7, putting $m'=n'^2a'^3$ (see Art. 19),

$$R=n'^2\frac{a'^3}{r'^3}\left(\tfrac{3}{2}r^2S^2-\tfrac{1}{2}r^2\right)+\frac{n'^2}{r'}\frac{a'^3}{r'^3}\left(\tfrac{5}{2}r^3S^3-\tfrac{3}{2}r^3S\right)+\dots.$$

Let v' be the true longitude of the Sun supposed to move in an elliptic orbit, and let the axis of X, which is rotating with the mean angular velocity of the Sun, point towards the Sun's mean place. If ϵ' be the angle, at time $t=0$, which this axis makes with the fixed line from which v' is reckoned, we shall have

$$X'=r'\cos(v'-n't-\epsilon'),\qquad Y'=r'\sin(v'-n't-\epsilon').$$

If now we neglect the solar eccentricity, these equations give

$$X'=r'=a',\qquad Y'=0,\qquad rS=(XX'+YY')/r'=X,$$

and the first term of R becomes $n'^2\left(\tfrac{3}{2}X^2-\tfrac{1}{2}r^2\right)$.

If therefore we neglect the terms beyond the first in R, that is, if we neglect terms which depend on the parallax of the Sun (retaining those dependent on the solar eccentricity), we shall have

$$R = n'^2\left(\tfrac{3}{2}X^2 - \tfrac{1}{2}r^2\right) + n'^2\left\{\tfrac{3}{2}\left(\frac{a'^3}{r'^3}r^2S^2 - X^2\right) - \tfrac{1}{2}r^2\left(\frac{a'^3}{r'^3} - 1\right)\right\}.$$

The second term of this expression then vanishes with the solar eccentricity. Moreover since r^2S^2 is always a quadratic function of X, Y, we can put

$$R = n'^2\left(\tfrac{3}{2}X^2 - \tfrac{1}{2}r^2\right) - \tfrac{1}{2}A'X^2 - B'XY - \tfrac{1}{2}C'Y^2 - \tfrac{1}{2}K'z^2,$$

in which A', B', C', K' are simple functions of the time depending on the solar elliptic motion.

Substituting, equations (12) become

$$\left.\begin{aligned}\ddot{X} - 2n'\dot{Y} - 3n'^2X + A'X + B'Y &= -\frac{\mu X}{r^3}\\ \ddot{Y} + 2n'\dot{X} \qquad\qquad + B'X + C'Y &= -\frac{\mu Y}{r^3}\\ \ddot{z} \qquad + n'^2 z \qquad\qquad + K'z &= -\frac{\mu z}{r^3}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (24),$$

in which those terms depending on the distance of the Sun are the only ones neglected. These equations form the basis of Adams' researches* into the connection between certain parts of the motions of the perigee and node and the constant part of $1/r$.

23. A further simplification is introduced by supposing the solar eccentricity and the latitude of the Moon neglected. Giving therefore A', B', C', K', z, zero values, the equations are reduced to the two

$$\left.\begin{aligned}\ddot{X} - 2n'\dot{Y} - 3n'^2X &= -\frac{\mu X}{r^3}\\ \ddot{Y} + 2n'\dot{X} \qquad\quad &= -\frac{\mu Y}{r^3}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

These play an important part in Hill's method of treating the lunar theory †. They are the equations of motion of a satellite disturbed by a body supposed to be of very great mass m', at a very great distance a', such that $m'/a'^3 = n'^2$ is a finite quantity. The disturbing body whose distance has just been supposed to be so great that $1/a'$ is negligible, is moving with uniform velocity in a circular orbit in the plane of motion of the satellite and is placed on the positive half of the moving X-axis.

The equations admit of a particular solution

$$Y = 0, \quad X = r = \text{const.} = (\mu/3n'^2)^{\frac{1}{3}}.$$

The Moon is then always on the axis of X, or in other words, is constantly in con-

* *Monthly Notices of the Royal Astronomical Society*, vol. XXXVIII., pp. 460—472.

† Chapter XI.

junction with the Sun. The motion is however unstable, a fact which can easily be obtained from the equations (25).

This is a particular case of a more general theorem mentioned below. (See Art. 30.)

24. It is not difficult to find expressions for the velocity when we neglect the solar eccentricity *only*. Since in this case, $rS = X$, $r' = a'$, we have, from Art. 22,

$$R = n'^2 (\tfrac{3}{2}X^2 - \tfrac{1}{2}r^2) + \frac{n'^2}{a'}(\tfrac{5}{2}X^3 - \tfrac{3}{2}r^2 X) + \dots$$

Hence R does not contain the time explicitly.

Multiply the equations (12) by $\dot{X}$, $\dot{Y}$, $\dot{z}$ and, after adding the results, integrate. We obtain

$$\dot{X}^2 + \dot{Y}^2 + \dot{z}^2 = \frac{2\mu}{r} + n'^2 (X^2 + Y^2) + 2R + \text{const.} \quad \text{.....................(26)},$$

giving the velocity referred to moving axes. (If we include the solar eccentricity, the term $-2\int \frac{\partial R}{\partial t} dt$ must be added to the right-hand side of (26)).

But we have $\quad \dot{X}^2 + \dot{Y}^2 + \dot{z}^2 = \dot{r}_1^2 + r_1^2 (\dot{\vartheta} - n')^2 + \dot{z}^2$

$$= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2n' (\dot{y}x - \dot{x}y) + n'^2 (x^2 + y^2).$$

Hence, since $\quad X^2 + Y^2 = x^2 + y^2,$

we obtain $\quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2\mu}{r} + 2n' (\dot{y}x - \dot{x}y) + 2R + \text{const.},$

giving the velocity in space, when the solar eccentricity is the only quantity neglected. This expression was first obtained by Jacobi*.

(iv) *The general Problem of Three Bodies.*

25. The developments given above refer to the motion of one body about a second when that of the third body about the second is supposed known. The problem of three bodies or rather of three particles, considered without any limitations, admits of a much more general treatment and moreover, when looked at from this point of view, is seen to possess certain properties in the form of first integrals which do not appear in the more restricted problem.

Let m_1, m_2, m_3 be the masses of the three bodies, r_{23}, r_{13}, r_{12} their mutual distances and $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$ their coordinates referred to rectangular axes, fixed in direction, through any origin. The nine equations of motion are

$$m_i \ddot{x}_i = \frac{\partial F}{\partial x_i}, \qquad m_i \ddot{y}_i = \frac{\partial F}{\partial y_i}, \qquad m_i \ddot{z}_i = \frac{\partial F}{\partial z_i}, \qquad (i = 1, 2, 3),$$

where, according to Art. 10, Chap. I.,

$$F = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}.$$

* *Comptes Rendus*, vol. III. pp. 59—61. *Collected Works*, vol. IV. pp. 37, 38.

Since $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$ and therefore

$$\frac{\partial}{\partial x_i}\frac{1}{r_{ij}} = -\frac{x_i - x_j}{r_{ij}^3} = -\frac{\partial}{\partial x_j}\frac{1}{r_{ij}}, \qquad (i, j = 1, 2, 3),$$

we get immediately, by addition of the equations of motion,

$$\Sigma m_i \ddot{x}_i = 0, \qquad \Sigma m_i \ddot{y}_i = 0, \qquad \Sigma m_i \ddot{z}_i = 0.$$

From these we obtain by integration

$$\Sigma m_i \dot{x}_i = a, \qquad \Sigma m_i \dot{y}_i = b, \qquad \Sigma m_i \dot{z}_i = c \text{ ....} (27),$$

where a, b, c are three constants. These constitute three first integrals of the equations of motion.

On integrating again, we have three further equations of the form

$$\Sigma m_i x_i = at + \text{const.},$$

or, eliminating a, b, c, three equations of the form[1]

$$\Sigma m_i x_i - t(\Sigma m_i \dot{x}_i) = \text{const.}, \text{......................} (28),$$

which are also three first integrals of the equation of motion.

Again, since

$$\left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}\right)\frac{1}{r_{ij}} = \frac{y_i x_j - x_i y_j}{r_{ij}^3} = -\left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}\right)\frac{1}{r_{ij}},$$

we obtain from the equations of motion,

$$\Sigma (x_i \ddot{y}_i - y_i \ddot{x}_i) = 0, \qquad \Sigma (y_i \ddot{z}_i - z_i \ddot{y}_i) = 0, \qquad \Sigma (z_i \ddot{x}_i - x_i \ddot{z}_i) = 0,$$

and therefore three more first integrals of the equations, of the form

$$\Sigma_i (x_i \dot{y}_i - y_i \dot{x}_i) = \text{const. ....} (29).$$

Finally, since F is a function of t only in so far as t occurs implicitly through the presence of the nine coordinates in F, we shall obtain, after multiplying the nine equations by their respective velocities, adding and integrating,

$$m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + m_3(\dot{x}_3^2 + \dot{y}_3^2 + \dot{z}_3^2) = 2F + \text{const.} \ldots (30),$$

a tenth first integral of the equations of motion.

26. At first sight the general integral of Energy (30) seems inconsistent with the expressions obtained in Arts. 12, 20, 24 which, when the solar eccentricity was not neglected, contained an unknown integral. It is to be remembered, however, that we have earlier supposed the motion of the Sun round the Earth to be known and to be expressible by known functions of the time. In so doing we have neglected a portion of the effect produced by the Moon on the motion of the Earth,—a portion which would produce an unknown integral in the expression for the relative Energy of the Sun. We have then divided the Energy, relative to the Earth, of the two bodies into two parts, one part being

that of the Moon and the other that of the Sun. In the motion of the Sun, the portion depending on the Moon is so small compared to its own great mass that we have neglected it, while, in the motion of the Moon, the same portion, being great compared to the mass of the Moon, cannot be neglected. In fact, if we denote, in the expression for the square of the Moon's velocity, this portion by $m'\phi$, where ϕ is a function of the coordinates and velocities, there will be, in the expression for the square of the Sun's velocity, a term $-M\phi$. When we add the Energy of the Sun to that of the Moon, ϕ will vanish identically.

27. The ten integrals found above are the only known integrals for the general problem of three bodies. It has been demonstrated further by M. Bruns*, that no other *algebraic* uniform integral can exist for *any* values of the masses. M. Poincaré† has extended this result, from a practical standpoint, by proving that if the ratios of two of the masses to the third are sufficiently small quantities, there does not exist any other *transcendental or algebraic* uniform integral. For the proofs of these theorems, which are based on considerations altogether outside the scope of this book, the reader is referred to the original memoirs.

It is evident that these ten integrals exist and are of the same form for any number of bodies attracting one another according to the Newtonian Law. The extension to this general case is made immediately, if we suppose i, j to receive the values $1, 2, \ldots p$, there being p bodies under consideration.

The ten integrals might have been written down from purely dynamical considerations. The first six integrals (27), (28) are known as those of the Centre of Mass and they express the facts that the linear momentum in any direction is constant and that the motion of the Centre of Mass is uniform and rectilinear. The three equations (29) are known as the integrals of areas. The dynamical equivalent is expressed by saying that the angular momentum round any line, fixed in direction, is constant. Equation (30) is that of Energy and expresses the fact that the sum of the Kinetic and Potential Energies is constant.

28. Let the three constants of angular momentum be h_1, h_2, h_3. The straight line whose direction cosines are proportional to h_1, h_2, h_3 is invariable in direction and consequently the plane perpendicular to it is so also. If we consider all the bodies of the Solar System without any reference to those outside, the plane determined in this way is known as the Invariable Plane of the Solar System. Laplace suggested that this plane might be used as a plane of reference to which the motions of the bodies might be referred. There are however several difficulties in the way.

* *Acta Mathematica*, vol. XI. p. 59.

† *Acta Mathematica*, vol. XIII. p. 264. Also *Mécanique Céleste*, vol. I. p. 253.

For further remarks on the Invariable Plane, see

E. J. Routh, *Rigid Dynamics*, Vol. I. Arts. 301–305.

Laplace, *Méc. Cél.* Book VI. Nos. 45, 46.

De Pontécoulant, *Système du Monde*, Vols. I. p. 455; II. p. 501; III. p. 528, p. 555.

Other references are given by Tisserand, *Méc. Cél.* Vol. I. p. 158.

29. We may consider any one of the equations of motion as replaced by two others, each of the first order. Let x be any coordinate, $\dot{x}$ its velocity. The equation

$$\frac{d^2x}{dt^2} = \frac{\partial F}{\partial x},$$

may be replaced by

$$\frac{d\dot{x}}{dt} = \frac{\partial F}{\partial x}, \quad \frac{dx}{dt} = \dot{x},$$

so that if there be p bodies we shall have $6p$ equations to determine $6p$ variables, namely, the coordinates and the velocities. By means of the ten integrals, it is theoretically possible to eliminate ten of the $6p$ variables and the resulting equations will contain $6p-10$ variables, or, in the case of three bodies, 8 variables. In general, it is found better to eliminate only six by means of the equations (27), (28), leaving in the case of three bodies, 12 variables between which four relations are known.

The literature on the general problem of three bodies dates chiefly from the researches of Lagrange. An account of these is given by F. Tisserand, *Mécanique Céleste*, Vol. I. Chap. VIII. and by O. Dziobek, *Die mathematischen Theorien der Planeten-Bewegungen*, pp. 80–82.

30. There are two special cases in which it is possible to integrate rigorously the equations of motion. They are obtained by supposing that the mutual distances of the three bodies remain constantly in the same ratio. In the first case, the three bodies are constantly in a straight line and each describes an ellipse with the common Centre of Mass as one focus. This motion is unstable. The result of Art. (23) is a special case of this. In the second case, the bodies always remain at the corners of an equilateral triangle of varying size.

These two problems are discussed by

F. Tisserand, *Méc. Cél.* Vol. I. Chap. VIII.

E. J. Routh, *Rigid Dynamics*, Vols. I. Art. 286, II. Arts. 108, 109.

In the former will be found further references to the papers on this subject. In the latter the stability of the second case is considered.

CHAPTER III.

UNDISTURBED ELLIPTIC MOTION.

31. THE subject of elliptic motion belongs properly to the problem of two bodies, the solution of which presents no difficulties: it is treated in most of the text-books on Dynamics. By the introduction of a certain angle (the eccentric anomaly), the relations between the coordinates and the time can be put into finite forms which, though useful for some purposes, are not convenient when we proceed to the problem of three bodies, this latter problem being generally treated, so far as the solar system is concerned, from the point of view of disturbed elliptic motion. In this case it becomes necessary to express most of the relations by means of series. These series, investigated mainly by means of Bessel's functions, will be given briefly in this Chapter. The subject will be divided into two parts, the first referring only to the properties of the elliptic *curve*, and the second containing applications of the results obtained in the first part, to elliptic *motion* about a centre of force in the focus.

(i) *Formulæ, Expansions and Theorems connected with the elliptic curve.*

32. Let C be the centre of an ellipse, E one focus, A the apse nearer to E, P any point on the ellipse, Q the corresponding point on the auxiliary circle and QPN the ordinate drawn perpendicular to CA.

Let $EP = r$ and let the angle $ACQ = \text{E}$, the angle $AEP = f$, and let w be defined by the equation

$$\frac{w}{2\pi} = \frac{\text{area } AEP}{\text{area of ellipse}}.$$

Then

f is called the true anomaly,

E „ „ eccentric „

w „ „ mean „

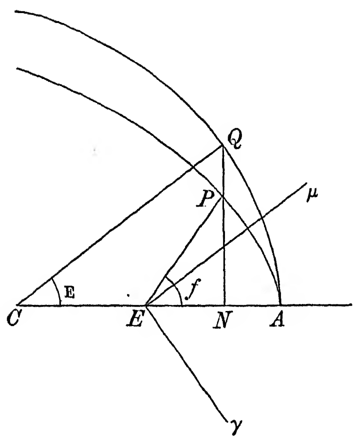

Fig. 3.

Let the major axis of the ellipse be $2a$, its eccentricity e and latus rectum l. From the well-known properties of corresponding points on an ellipse and its auxiliary circle, and from fig. 3, it is evident that the following relations hold:—

$$r = a(1 - e\cos E) = l/(1 + e\cos f) = a(1 - e^2)/(1 + e\cos f) \text{......(1)},$$

$$r\cos f = a(\cos E - e), \qquad r\sin f = a\sqrt{1-e^2}\sin E \text{ ....(2)},$$

$$w = E - e\sin E, \qquad \tan\tfrac{1}{2}f = \frac{\sqrt{1+e}}{\sqrt{1-e}}\tan\tfrac{1}{2}E \text{.................(3)}.$$

Also, since a, e, l remain constant and r, f, E, w vary with the position of P, we obtain easily

$$r^2\frac{df}{dw} = a^2\sqrt{1-e^2}, \qquad \frac{dr}{dw} = \frac{ae}{\sqrt{1-e^2}}\sin f, \qquad \frac{dE}{dw} = \frac{a}{r} \text{.........(4)}.$$

There are two problems to be considered. The first consists in expressing certain functions of w and r in series proceeding according to sines and cosines of multiples of f and powers of e; the second consists in expressing certain functions of r and f by similar series in terms of w and e. These series will be investigated first in an elementary way and then by means of Bessel's functions.

33. *To obtain series for r, w in terms of f.*

Define λ by the equation

$$e = 2\lambda/(1+\lambda^2), \text{ or } \lambda = (1-\sqrt{1-e^2})/e = e/(1+\sqrt{1-e^2}).$$

We then have by (1) $$\frac{r}{a} = \frac{(1-\lambda^2)^2}{1+\lambda^2}\,\frac{1}{1+\lambda^2+2\lambda\cos f}.$$

Putting for a moment $2\cos f = x^1 + x^{-1}$, and therefore $2\cos if = x^i + x^{-i}$, we easily obtain

$$\frac{r}{a} = \frac{1-\lambda^2}{1+\lambda^2}\left(\frac{1}{1+\lambda x} - \frac{\lambda x^{-1}}{1+\lambda x^{-1}}\right),$$

and thence by expansion, since x is a complex quantity whose modulus is unity,

$$\frac{r}{a} = (1-e^2)^{\frac{1}{2}}\{1 - 2\lambda\cos f + \ldots + (-1)^i\, 2\lambda^i \cos if + \ldots\} \quad \ldots\ldots\ldots (5).$$

From this equation, after expanding λ in powers of e, we obtain the required expansion of r.

Again, from (5) we have the identity

$$(1+e\cos f)^{-1} = (1-e^2)^{-\frac{1}{2}}\{1 - 2\lambda\cos f + \ldots + (-1)^i\, 2\lambda^i\cos if + \ldots\}.$$

Since the series on the right is supposed to be convergent, we may differentiate this equation with respect to e; we then get, after multiplying by e,

$$-\frac{e\cos f}{(1+e\cos f)^2} = \frac{e^2}{(1-e^2)^{\frac{3}{2}}}\left\{1 + 2\sum_1^\infty (-1)^i\lambda^i\cos if\right\}$$
$$+ \frac{2e}{\sqrt{1-e^2}}\,\frac{d\lambda}{de}\sum_1^\infty (-1)^i\, i\lambda^{i-1}\cos if.$$

Adding this to the previous equation we obtain, since $d\lambda/de = \lambda/e\sqrt{1-e^2}$,

$$\frac{1}{(1+e\cos f)^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}}\left\{1 + 2\sum_1^\infty (-1)^i\lambda^i\cos if\right\} + \frac{2}{1-e^2}\sum_1^\infty (-1)^i\, i\lambda^i\cos if$$
$$= (1-e^2)^{-\frac{3}{2}}\left\{1 + 2\sum_1^\infty (-1)^i\lambda^i(1+i\sqrt{1-e^2})\cos if\right\}.$$

But we have, by (4),

$$\frac{dw}{df} = \frac{r^2}{a^2\sqrt{1-e^2}} = \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos f)^2}.$$

Substituting for $(1+e\cos f)^{-2}$ its expansion just found and integrating, we obtain, since w and f are zero together,

$$w = f - 2\lambda(1+\sqrt{1-e^2})\sin f + \ldots + \frac{2}{i}(-1)^i\lambda^i(1+i\sqrt{1-e^2})\sin if + \ldots \ldots (6).$$

Putting for λ its value and expanding in powers of e, we deduce the required expansion of w. This is, as far as the order e^4,

$$w = f - 2e\sin f + (\tfrac{3}{4}e^2 + \tfrac{1}{8}e^4)\sin 2f - \tfrac{1}{3}e^3 \sin 3f + \tfrac{5}{32}e^4 \sin 4f - \ldots\ldots(6').$$

34. *To obtain f, r in terms of w.*

Since we have supposed that convergent series are possible, we can obtain f in terms of w by reversing the series (6′) using $f = w$ as a first approximation. By this proceeding we shall get, as far as the order e^4,

$$f = w + (2e - \tfrac{1}{4}e^3)\sin w + (\tfrac{5}{4}e^2 - \tfrac{11}{24}e^4)\sin 2w + \tfrac{13}{12}e^3 \sin 3w + \tfrac{103}{96}e^4 \sin 4w + \ldots(7).$$

Also as $r = a(1 - e\cos E)$, $w = E - e\sin E$, we can apply Lagrange's theorem* to the expansion of $\cos E = \cos(w + e\sin E)$ and thus obtain

$$\cos E = \cos w - \sum_1^\infty \frac{e^p}{p!}\frac{d^{p-1}}{dw^{p-1}}(\sin^p w \sin w).$$

From this we have, after replacing powers of $\sin w$ by sines of multiples of w, as far as the order e^4,

$$\frac{r}{a} = 1 + \frac{e^2}{2} - (e - \tfrac{3}{8}e^3)\cos w - (\tfrac{1}{2}e^2 - \tfrac{1}{3}e^4)\cos 2w - \tfrac{3}{8}e^3 \cos 3w - \tfrac{1}{3}e^4 \cos 4w - \ldots (8).$$

Corollary. Since $a^2/r^2 = df/dw$, we can immediately obtain from (7) the expansion of a^2/r^2.

35. From these expansions for f, r we can deduce those of any functions of f, r in terms of w. As however all these expansions (except perhaps that of f) are obtained much more easily by the use of other methods, those outlined in this Article will not be further developed. For a fuller discussion of the methods of Arts. 33, 34, see Tait and Steele, *Dynamics of a Particle*, Arts. 162–167.

36. An expansion for f in terms of w, exhibiting the general term of the series by means of a simple expression, has been given by S. S. Greatheed†. The process may be shortly sketched as follows. Let

$$\tan\frac{f_0}{2} = \frac{\sqrt{1+e}}{\sqrt{1-e}}\tan\frac{w}{2},$$

then by Lagrange's theorem applied to equations (3) we have

$$f = f_0 + \sum_1^\infty \frac{e^p}{p!}\frac{d^{p-1}}{dw^{p-1}}\left(\frac{df_0}{dw}\sin^p w\right).$$

Also, differentiating the expression for f_0 we have

$$\frac{df_0}{dw} = \frac{\sqrt{1-e^2}}{1 - e\cos w} = 2\sum_0^\infty \lambda^i \cos iw, \ \text{(where } \lambda^0 = \tfrac{1}{2}\text{)}$$

and‡

$$\sin^p w = \frac{1}{2^p}\sum_0^p (-1)^q \frac{p!}{q!(p-q)!}\cos\left\{(p-2q)w - p\frac{\pi}{2}\right\},$$

* Williamson, *Diff. Cal.* Chap. VII.

† *Camb. Math. Jour.*, 1st Ed. Vol. I. (1839), pp. 208—211.

‡ E. W. Hobson, *Trigonometry*, Art. 52. The formulæ there numbered (44), (45) can be combined into this form.

giving

$$\frac{d^{p-1}}{dw^{p-1}}\left(\frac{df_0}{dw}\sin^p w\right)=\frac{1}{2^p}\sum_{q=0}^{p}\sum_{i=0}^{\infty}(-1)^q\frac{p\,!}{q\,!\,(p-q)\,!}\lambda^i\{(p-2q+i)^{p-1}\sin(p-2q+i)\,w$$
$$+(p-2q-i)^{p-1}\sin(p-2q-i)\,w\}.$$

Substituting this in the equation for f and finding the coefficient of $\sin jw$ in the resulting expression, Greatheed finally arrives at the symbolic formula for f:

$$w+2\sum_{j=1}^{\infty}\left\{\lambda^j\exp.\frac{je}{2}(\lambda^{-1}-\lambda)+\lambda^{-j}\exp.-\frac{je}{2}(\lambda^{-1}-\lambda)\right\}\frac{\sin jw}{j}.$$

In order to obtain the coefficient of $\sin jw$, the expression for it given by this formula must be first developed in powers of λ as it stands, all negative powers of λ rejected and the terms of the order λ^0 divided by 2*.

Cayley† has extended this result in a general manner to the expansion of any function of r and f.

Expansions by means of Bessel's functions.

37. The following formulæ will be found‡ in any treatise on Bessel's functions, i being a real integer:—

$$J_i(x)=\frac{1}{\pi}\int_0^{\pi}\cos(i\phi-x\sin\phi)\,d\phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$J_i(x)=\frac{x^i}{2^i i\,!}\left[1-\frac{x^2}{2(2i+2)}+\ldots+\frac{(-1)^q x^{2q}}{2^{2q}q\,!\,(i+1)(i+2)\ldots(i+q)}+\ldots\right]\ldots(10),$$

$$\frac{x}{2}\{J_{i-1}(x)+J_{i+1}(x)\}=iJ_i(x),\quad \tfrac{1}{2}\{J_{i-1}(x)-J_{i+1}(x)\}=\frac{d}{dx}J_i(x)\ldots\ldots(11),$$

where $J_i(x)$ is the Bessel's function of the first kind. In the applications to be made here, i is a positive integer and x a real quantity sufficiently small for series in powers of x to be convergent. If i be negative we have immediately from (9), $J_i(x)=J_{-i}(-x)$.

38. *To expand* $\cos jE$, $\sin jE$ *in terms of* w.

From (3) we may assume that $\cos jE$, $\sin jE$ will be respectively expansible in cosines and sines of multiples of w. Let

$$\cos jE=\Sigma_i A_i\cos iw,\qquad \sin jE=\Sigma_i B_i\sin iw,$$

where j, i are positive integers. Then, by Fourier's theorem,

$$A_i\frac{\pi}{2}=\int_0^{\pi}\cos jE\cos iw\,dw.$$

* See note by Cayley on the expansion of this formula in *Quart. Math. Jour.* Vol. II., pp. 229—232. *Coll. Works*, Vol. III. pp. 139—142.

† *Camb. Math. Jour.* 1st Ed., Vol. III. pp. 162—167. *Coll. Works*, Vol. I. pp. 19-24.

‡ *E.g.* Todhunter, Chapters XXX. XXXI. It will not here be necessary to suppose any knowledge of Bessel's functions, beyond the assumptions that the series are possible and that they converge. If we define $J_i(x)$ by (9) the results (10), (11) can be found by a few simple operations.

Integrating by parts,

$$A_i = \frac{2}{\pi}\left[\frac{1}{i}\cos jE \sin iw\right]_{w=0}^{\pi} + \frac{2j}{i\pi}\int_0^{\pi} \sin iw \sin jE\,dE$$

$$= 0 + \frac{2j}{i\pi}\int_0^{\pi} \sin(iE - ie\sin E)\sin jE\,dE, \qquad \text{by (3)},$$

$$= \frac{j}{i\pi}\int_0^{\pi}\cos\{(i-j)E - ie\sin E\}\,dE - \frac{j}{i\pi}\int_0^{\pi}\cos\{(-i-j)E - (-ie)\sin E\}\,dE$$

$$= \frac{j}{i}J_{i-j}(ie) + \frac{j}{-i}J_{-i-j}(-ie), \qquad \text{by (9)},$$

except when $i = 0$. For the determination of A_0 we have

$$A_0\pi = \int_0^{\pi}\cos jE\,dw = \int_0^{\pi}(1 - e\cos E)\cos jE\,dE = -\frac{e}{2}\pi \text{ or } 0,$$

according as j is equal or unequal to unity.

If therefore we allow i to receive negative as well as positive values, we obtain

$$\cos jE = \sum_{i=-\infty}^{\infty}\frac{j}{i}J_{i-j}(ie)\cos iw \qquad (12),$$

in which $$\frac{j}{0}J_{-j}(0) = -\frac{e}{2} \text{ or } 0,$$

according as j is equal or unequal to unity.

In an exactly similar way we may find

$$B_i = \frac{1}{i}J_{i-j}(ie) - \frac{1}{-i}J_{-i-j}(-ie),$$

and $$\sin jE = \sum_{i=-\infty}^{\infty}\frac{j}{i}J_{i-j}(ie)\sin iw \qquad (13),$$

there being no constant term. From the results (12), (13) we can get most of the expansions required.

39. *To expand r, $r\cos f$, $r\sin f$, r^{-1}, r^{-2} in terms of w.*

Putting $j = 1$ in the two results just obtained and substituting for $\cos E$ and $\sin E$ in (1), (2), we deduce

$$\frac{r}{a} = 1 - \sum_{-\infty}^{\infty}\frac{e}{i}J_{i-1}(ie)\cos iw \qquad (14),$$

$$\left.\begin{aligned} \frac{r}{a}\cos f &= -e + \sum_{-\infty}^{\infty}\frac{1}{i}J_{i-1}(ie)\cos iw, \\ \frac{r}{a}\sin f &= \sqrt{1-e^2}\sum_{-\infty}^{\infty}\frac{1}{i}J_{i-1}(ie)\sin iw \end{aligned}\right\} \qquad (15).$$

Since $J_{-i}(-ie)=J_i(ie)$ and since for $i=0$ we have

$$\frac{1}{0}J_{-1}(0)=-\frac{e}{2},$$

we deduce, after the application of the formulæ (11),

$$\frac{r}{a}=1+\frac{e^2}{2}-\sum_1^\infty \frac{2e}{i^2}\frac{dJ_i(ie)}{de}\cos iw \qquad\text{(14')},$$

$$\left.\begin{aligned} r\cos f &= a\left[-\tfrac{3}{2}e+\sum_1^\infty \frac{2}{i^2}\frac{dJ_i(ie)}{de}\cos iw\right] \\ r\sin f &= a\sqrt{1-e^2}\left[\sum_1^\infty \frac{2}{ie}J_i(ie)\sin iw\right]\end{aligned}\right\} \qquad\text{(15')}.$$

Again, by (4), (3), (2), we have

$$\frac{a}{r}=\frac{dE}{dw}=1+\frac{d}{dw}(e\sin E)=1+\frac{d}{dw}\left(\frac{er\sin f}{a\sqrt{1-e^2}}\right),$$

and therefore from (15'),

$$\frac{a}{r}=1+2\sum_1^\infty J_i(ie)\cos iw \qquad\text{(16)}.$$

Further, since $r/a=1-e\cos E$, we have

$$\frac{d}{dw}\left(\frac{r^2}{a^2}\right)=2\frac{r}{a}\frac{d}{dw}\left(\frac{r}{a}\right)=2\frac{r}{a}\frac{dE}{dw}e\sin E=2e\sin E, \qquad \text{by (1), (4);}$$

and since (15') gives the expansion of $r\sin f=a\sqrt{1-e^2}\sin E$, we obtain

$$\frac{d}{dw}\left(\frac{r^2}{a^2}\right)=\sum_1^\infty \frac{4}{i}J_i(ie)\sin iw.$$

Integrating, $$\frac{r^2}{a^2}=\text{const.}-\sum_1^\infty \frac{4}{i^2}J_i(ie)\cos iw.$$

But since $$r^2/a^2=1-2e\cos E+\tfrac{1}{2}e^2+\tfrac{1}{2}e^2\cos 2E;$$

and since it was shown in the last Article that the constant part of $\cos E$, when expanded in terms of w, is $-e/2$ and that the similar part of $\cos 2E$ is 0, the above equation shows that the constant part of r^2/a^2, when so expanded, is $1+3e^2/2$. Hence

$$\frac{r^2}{a^2}=1+\tfrac{3}{2}e^2-\sum_1^\infty \frac{4}{i^2}J_i(ie)\cos iw \qquad\text{(17)}.$$

Corollary i. From (1) and (4) we have

$$\cos f=\frac{1-e^2}{e}\frac{a}{r}-\frac{1}{e}, \quad \sin f=\frac{\sqrt{1-e^2}}{ae}\frac{dr}{dw}.$$

Whence, by using the developments (16), (14'), we can immediately deduce those of $\cos f$, $\sin f$.

The difference between the true and mean anomalies is called the *Equation of the Centre.* Denote it by Eq.

Corollary ii. If α be any angle, we have, since $f = w + \text{Eq.}$,

$$\sin(\alpha + \text{Eq.}) = \sin f \cos(\alpha - w) + \cos f \sin(\alpha - w).$$

By means of Cor. i. we can then get the development of $\sin(\alpha + \text{Eq.})$.

40. It will be noticed that the coefficient of $\sin iw$ or $\cos iw$ (i positive) is always of the form $\alpha_0 e^i + \alpha_1 e^{i+2} + \alpha_2 e^{i+4} + \ldots$, ($\alpha_0, \alpha_1, \alpha_2 \ldots$ numerical quantities). That this must be so in the expansions of all functions of r, f of the forms treated here, is sufficiently evident from Art. 32. Hence if we are considering any term whose argument is iw, we know immediately that the lowest power of e contained in the coefficient is not less than e^i. This fact has an important bearing when we come to develope the disturbing function.

41. In the development of the disturbing function it is important to obtain expansions for $r^p \cos qf$, $r^p \sin qf$ (p being any positive or negative integer and q any integer including zero) in terms of w. These could be obtained from the expansions given in Art. 39 by multiplication of series. Such a process would be somewhat tedious when many terms are required. On pp. 163–179 of the *Fundamenta**, Hansen obtains the expansions by finding the finite expressions corresponding to each value of p and q required—for $r^p \cos qf$ in terms of positive or negative powers of r, and for $r^p \sin qf$ in terms of the differentials of the same with respect to w. That this is possible is evident from the expressions for $\cos f$ and $\sin f$ given in Cor. i. of Art. 39. He then obtains a general formula giving the coefficients of the development of r^p in terms of those of the developments of r^2 and r^{-2}. The coefficients of r^2 are obtained as in Art. 39 and those of r^{-2} as in the Cor. of Art. 34.

In a later work†, he has considered them in a much more general manner and has obtained expressions for the coefficients of the development of $r^p \exp. qf\sqrt{-1}$ in powers of $\exp. w\sqrt{-1}$, by means of Fourier's theorem. The definite integral corresponding to the coefficient of $\exp. iw\sqrt{-1}$, is evaluated and a general expression which is the least cumbrous to expand of any given up to the present time, is obtained for the coefficient. This is even true of the case $p = -2$, $q = 0$, which by a simple integration gives the development of f.

42. The literature on elliptic expansions up to 1862 has been collected by Cayley in a report *On the progress of the solution of certain problems in Dynamics*‡. Later developments and references are to be found in Tisserand, *Méc. Cél.*, Vol. I. chaps. XIV., XV.

43. The following Theorem and Corollary will be required later.

Let F, G, H be three functions of which F, G are developable in cosines (or sines) and H in sines (or cosines) of a series of angles of the general form $\beta t + \beta'$. The function

$$\Gamma = F + G\left(\frac{r\cos f}{a} + \tfrac{3}{2}e\right) + H\frac{r\sin f}{a}$$

* *Fundamenta Nova Investigationis Orbitae verae quam Luna perlustrat.* Auctore P. A. Hansen. Gotha, 1838. This work will be referred to throughout as the *Fundamenta*.

† 'Entwicklung des Products einer Potenz des Radius-Vectors mit dem Sinus oder Cosinus eines Vielfachen der wahren Anomalie etc.' *Abh. d. K. Sächs. Ges. zu Leipzig*, Vol. II. pp. 183—281.

‡ *Brit. Assoc. Reports*, 1862. *Coll. Works*, Vol. IV. pp. 513—593.

can be developed into a series of the form

$$\Gamma = \Sigma \sum_{i=-\infty}^{\infty} \alpha_i \frac{\cos}{\sin}(iw + \beta t + \beta');$$

also, when the coefficients α_0, α_1, α_{-1} *have been found, all the other coefficients* α_i *can be obtained by a simple process.*

Suppose that F, G are developable in cosines and H in sines of angles of the form $\beta t + \beta'$. Let

$$-\tfrac{1}{2} G = \Sigma B \cos(\beta t + \beta'), \quad H \frac{\sqrt{1-e^2}}{2e} = \Sigma B' \sin(\beta t + \beta'),$$

in which B, B' are the typical coefficients corresponding to the argument $\beta t + \beta'$. Let

$$-\frac{2}{i^2} J_i(ie) = R_i.$$

The formulæ (15′) may be written

$$\frac{r\cos f}{a} + \tfrac{3}{2}e = -\sum_1^{\infty} \frac{dR_i}{de}\cos iw, \quad \frac{r\sin f}{a} = -\frac{\sqrt{1-e^2}}{e}\sum_1^{\infty} iR_i \sin iw \ldots(18);$$

also from (17) we have

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 + 2\sum_1^{\infty} R_i \cos iw \ldots\ldots\ldots\ldots(19).$$

This last equation will be required in Chap. X.

Substituting in the expression for Γ we obtain

$$\Gamma - F = 2\Sigma B\cos(\beta t + \beta') \cdot \sum_{i=1}^{\infty} \frac{dR_i}{de}\cos iw - 2\Sigma B' \sin(\beta t + \beta') \cdot \sum_{i=1}^{\infty} iR_i \sin iw$$

$$= \Sigma \sum_{i=1}^{\infty} \left[\left(B\frac{dR_i}{de} + B' iR_i\right)\cos(iw + \beta t + \beta') + \left(B\frac{dR_i}{de} - B' iR_i\right)\cos(-iw + \beta t + \beta')\right].$$

If now we put $R_{-i} = R_i$, $R_0 = 0$, this may be written

$$\Gamma - F = \Sigma \sum_{i=-\infty}^{\infty} \alpha_i \cos(iw + \beta t + \beta'), \quad (i = 0 \text{ excluded})$$

where
$$\alpha_i = B\frac{dR_i}{de} + B' iR_i.$$

If we put also

$$F = \Sigma \alpha_0 \cos(\beta t + \beta'),$$

we obtain
$$\Gamma = \Sigma \sum_{i=-\infty}^{\infty} \alpha_i \cos(iw + \beta t + \beta')$$

for all values of i.

By the definition of α_i we have, since $R_i = R_{-i}$,

$$\alpha_1 = B\frac{dR_1}{de} + B'R_1, \quad \alpha_{-1} = B\frac{dR_1}{de} - B'R_1;$$

whence $$B = (\alpha_1 + \alpha_{-1})/2\frac{dR_1}{de}, \quad B' = (\alpha_1 - \alpha_{-1})/2R_1.$$

Substituting these values of B, B' in the expressions for α_i, α_{-i}, we obtain

$$\alpha_i = \tfrac{1}{2}\alpha_1\left[\frac{\frac{dR_i}{de}}{\frac{dR_1}{de}} + i\frac{R_i}{R_1}\right] + \tfrac{1}{2}\alpha_{-1}\left[\frac{\frac{dR_i}{de}}{\frac{dR_1}{de}} - i\frac{R_i}{R_1}\right],$$

$$\alpha_{-i} = \tfrac{1}{2}\alpha_{-1}\left[\frac{\frac{dR_i}{de}}{\frac{dR_1}{de}} + i\frac{R_i}{R_1}\right] + \tfrac{1}{2}\alpha_1\left[\frac{\frac{dR_i}{de}}{\frac{dR_1}{de}} - i\frac{R_i}{R_1}\right].$$

When therefore α_1, α_{-1} have been found for all the different arguments $\beta t + \beta'$, the coefficients α_i, α_{-i} can be found without any trouble. This simple method of obtaining the coefficients in the product of two series, saves Hansen much labour in performing his developments.

Corollary. From the last two equations we deduce immediately

$$(\alpha_i - \alpha_{-i})/iR_i = (\alpha_1 - \alpha_{-1})/R_1.$$

44. *When the plane of the ellipse is inclined to the plane of reference,* expansions for the longitude in this latter plane and for the latitude above it will be required.

Let M (fig. 4) be the position on the unit sphere corresponding to the point P (fig. 3) whose true anomaly is f. Let $M\Omega$ be the position of the

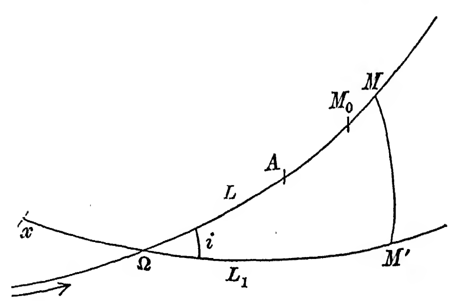

Fig. 4.

plane of the orbit and draw MM' perpendicular to xM' the plane of reference. Then, according to the notation of Chapter II., $v = xM'$, $s = \tan M'M$. Let $x\Omega = \theta$, $\Omega M = L$, $\Omega M' = L_1$, $M'\Omega M = i$* and therefore $L_1 = v - \theta$.

* The letter i is frequently used, as in the previous articles, to denote any integer: the additional use of the letter to denote this angle will cause no confusion.

45. From the right-angled triangle $MM'\Omega$ we have

$$\tan L_1 = \cos i \tan L.$$

Putting $\iota = \sqrt{-1}$, we can write this

$$\frac{e^{2L_1\iota}-1}{e^{2L_1\iota}+1} = \cos i\,\frac{e^{2L\iota}-1}{e^{2L\iota}+1},$$

and therefore
$$e^{2L_1\iota} = e^{2L\iota}\,\frac{1+\tan^2\frac{i}{2}\,e^{-2L\iota}}{1+\tan^2\frac{i}{2}\,e^{2L\iota}}.$$

Taking logarithms and expanding, we obtain

$$2L_1\iota = 2p\pi\iota + 2L\iota - \tan^2\frac{i}{2}(e^{2L\iota}-e^{-2L\iota}) + \tfrac{1}{2}\tan^4\frac{i}{2}(e^{4L\iota}-e^{-4L\iota}) - \ldots,$$

where p is an integer. Since $L_1 = L$ when i is zero, we have $p = 0$. Hence

$$L_1 = L - \tan^2\frac{i}{2}\sin 2L + \tfrac{1}{2}\tan^4\frac{i}{2}\sin 4L - \tfrac{1}{3}\tan^6\frac{i}{2}\sin 6L + \ldots.$$

Let now the angular distance from the apse to the node Ω be $\varpi - \theta$. We have then

$$L = f + \varpi - \theta = w + \varpi - \theta + \text{Eq.} = \eta_0 + \text{Eq.},$$

where $\eta_0 = w + \varpi - \theta$. Substituting for L this value and for L_1 its value $v - \theta$, we obtain

$$v = f + \varpi - \tan^2\frac{i}{2}\sin(2\eta_0 + 2\,\text{Eq.}) + \tfrac{1}{2}\tan^4\frac{i}{2}\sin(4\eta_0 + 4\,\text{Eq.}) - \ldots (20).$$

The terms involving i and constituting the difference between the longitude in the orbit and that in the plane of reference are known as the *Reduction.*

We can expand

$$\sin(2\eta_0 + 2\,\text{Eq.}) = \sin 2\eta_0 \cos 2\,\text{Eq.} + \cos 2\eta_0 \sin 2\,\text{Eq.}$$

by means of the formulæ given in Cor. ii., Art. 39. Let

$$\tan i = \gamma.$$

Then
$$\tan^2\frac{i}{2} = (2 + \gamma^2 - 2\sqrt{1+\gamma^2})/\gamma^2 = \tfrac{1}{4}\gamma^2 - \tfrac{1}{8}\gamma^4 + \tfrac{5}{64}\gamma^6 - \ldots.$$

In the case of our Moon, γ is a small quantity of the same order as e: it will, therefore, not be necessary to calculate a large number of terms in the Reduction, when the latter is expanded in powers of e and γ.

46. To obtain s, the tangent of the latitude, we have

$$\sin M'M = \sin i \sin L,$$

and therefore
$$s = \frac{\sin i \sin L}{\sqrt{1 - \sin^2 i \sin^2 L}} = \frac{\gamma \sin L}{\sqrt{1 + \gamma^2 \cos^2 L}},$$

giving
$$s = \gamma \sin L - \tfrac{1}{2}\gamma^3 \sin L \cos^2 L + \tfrac{3}{8}\gamma^5 \sin L \cos^4 L - \dots,$$

or
$$s = \gamma \sin(\eta_0 + \text{Eq.}) - \tfrac{1}{8}\gamma^3 \{\sin(3\eta_0 + 3\text{ Eq.}) + \sin(\eta_0 + \text{Eq.})\} + \dots (21).$$

Of these terms in s, the first is the most important and the method of finding it has been given in Cor. ii., Art. 39. The other terms can be easily calculated; since they are multiplied by γ^3 at least, it will not be necessary to take many of them.

47. It will be noticed that there is a connection between the index of γ and the multiple of η_0 similar to that between the index of e and the multiple of w. In longitude we have even multiples of η_0 and even powers of γ; in latitude, odd multiples of η_0 and odd powers of γ. In both cases, the lowest power of γ which occurs in the coefficient of $\sin i\eta_0$ or $\cos i\eta_0$, is γ^i.

(ii) *Elliptic Motion.*

48. When we neglect the disturbing action of the Sun, the forces on the Moon, relative to the Earth, are reduced to μ/r^2 acting inwards along the radius vector. Such a force is known to produce motion in an ellipse of which one focus is occupied by the Earth. We shall not here solve the problem which is merely that of two bodies, but assume that the solution has been completed; all that then remains is to fix the constants to be used.

Let (fig. 3) $E\gamma$ be a fixed line from which we may reckon angles. Let $\gamma EA = \varpi$, and let ϵ be the angle which a uniformly revolving radius vector $E\mu$ makes with $E\gamma$ at time $t = 0$. Let the time of a complete revolution of this vector be $2\pi/n$. Then $\gamma E\mu = nt + \epsilon$ and $AE\mu = nt + \epsilon - \varpi$. But since equal areas are described in equal times, we have, by the definition of w in Art. 32, $AE\mu = w$. Hence

$$\text{Mean anomaly} = w = nt + \epsilon - \varpi.$$

Let $2a$ be the major axis and e the eccentricity. Then we have the following well-known results:—

$$\mu = n^2 a^3, \quad (\text{Velocity})^2 = \frac{2\mu}{r} - \frac{\mu}{a},$$

Twice the area described in a unit of time $= na^2\sqrt{1 - e^2}$,

and n (or a), e, ϵ, ϖ may be taken as the four constants of integration.

49. When the plane of motion is inclined to the plane of reference we require two more constants. Let them be those defined in Art. 44, namely, i the inclination and θ the angular distance of the line of intersection of the two planes from the fixed line Ex. In this case the line $E\gamma$ is taken to coincide with Ex, so that ϖ, ϵ are reckoned from x along the fixed plane to Ω (the *ascending node*), and then along the plane of the orbit. Let M_0 (fig. 4) be the position, on the unit sphere, of $E\mu$ (fig. 3). Then

$$\varpi = x\Omega + \Omega A, \quad nt + \epsilon = x\Omega + \Omega M_0,$$

and
$$\Omega M_0 = \eta_0 = w + \varpi - \theta = nt + \epsilon - \theta.$$

The last angle is known as the *mean argument of the latitude*. The constants introduced by the three equations which determine undisturbed elliptic motion in space, are a (or n), e, ϵ, ϖ, θ, i (or γ).

These six constants are called the *Elements* of the ellipse. The meaning to be attached to the word 'Element' will be extended in Chapter V.

50. From the results of Arts. 34, 45, 46, we obtain the following values of v, r, s in terms of the time, for the solution of the equations (A) of Chapter II. when we neglect R, as far as the 3rd order of the small quantities e, γ:

$$\left.\begin{aligned} v &= nt + \epsilon + (2e - \tfrac{1}{4}e^3)\sin w + \tfrac{5}{4}e^2 \sin 2w + \tfrac{13}{12}e^3 \sin 3w + \dots \\ &\quad - \tfrac{1}{4}\gamma^2 \sin 2\eta_0 - \tfrac{1}{2}e\gamma^2 \sin(w - 2\eta_0) - \tfrac{1}{2}e\gamma^2 \sin(w + 2\eta_0) + \dots, \\ r/a &= 1 + \tfrac{1}{2}e^2 - (e - \tfrac{3}{8}e^3)\cos w - \tfrac{1}{2}e^2 \cos 2w - \tfrac{3}{8}e^3 \cos 3w - \dots, \\ s &= (1 - e^2 - \tfrac{1}{8}\gamma^2)\,\gamma \sin \eta_0 + e\gamma \sin(w - \eta_0) + e\gamma \sin(w + \eta_0) \\ &\quad + \tfrac{1}{8}e^2\gamma \sin(2w - \eta_0) + \tfrac{9}{8}e^2\gamma \sin(2w + \eta_0) - \tfrac{1}{8}\gamma^3 \sin 3\eta_0 + \dots \end{aligned}\right\} \dots (22),$$

where
$$w = nt + \epsilon - \varpi, \quad \eta_0 = nt + \epsilon - \theta.$$

51. It only remains to connect the constants a, h of Art. 13 with those used here. We have found in equation (2), Chapter II., neglecting R,

$$\text{Square of velocity in orbit} = 2\mu/r - \mu/a.$$

This being the same expression as that given in Art. 48, a has the same meaning in both cases, namely, the semi-major axis of the orbit. We have also by Art. 12,

$$\tfrac{1}{2}h = \tfrac{1}{2}r_1^2\dot{v} = \text{rate of description of areas in the plane of reference,}$$

whence

$$\tfrac{1}{2}\frac{h}{\cos i} = \text{rate of description of areas in the plane of the orbit,}$$

$$= \tfrac{1}{2}na^2\sqrt{1 - e^2}, \text{ by Art. 48.}$$

Hence
$$h = \frac{na^2\sqrt{1 - e^2}}{\sqrt{1 + \gamma^2}} \dots\dots\dots\dots (23).$$

This value refers to undisturbed motion only.

52. The solution of equations (11) of Chapter II. when $F=\mu/r$, may, since $u_1=u\sqrt{1+s^2}$, be put into the form

$$u_1=\frac{\sqrt{1+s^2}}{l}\{1+e\cos f\}=\frac{1}{l}\{\sqrt{1+s^2}+e_1\cos(v-\varpi_1)\},$$

$$s=\gamma\sin(v-\theta),\quad \frac{dt}{dv}=\frac{1}{hu_1^2},$$

where $\tan(\varpi_1-\theta)=\sqrt{1+\gamma^2}\tan(\varpi-\theta),\quad e_1=e\sqrt{1+\gamma^2\sin^2(\varpi-\theta)}.$

For we have, by the figure of Art. 44, since $M'M=v$,

$$e\sqrt{1+s^2}\cos f=\frac{e\cos\{L-(\varpi-\theta)\}}{\cos U}=e\cos(\varpi-\theta)\frac{\cos L}{\cos U}+e\sin(\varpi-\theta)\frac{\sin L}{\sin U}$$

$$=e\cos(\varpi-\theta)\cos L_1+e\sin(\varpi-\theta)\frac{\sin L_1}{\cos i};$$

whence, after substituting for e and ϖ in terms of e_1 and ϖ_1 and putting $L_1=v-\theta$, we get the required result. If AA_1, drawn perpendicular to the plane of the orbit, intersect xy in A_1, we easily obtain $\varpi_1=xA_1$.

To obtain u_1 in terms of v, we expand $\sqrt{1+s^2}$ by the binomial theorem and, after substituting for s its value $\gamma\sin(v-\theta)$, express these terms in cosines of multiples of $2(v-\theta)$. To obtain t in terms of v, we can expand $1/u_1^2$ by means of the formulæ of Art. 33. For we have

$$\frac{1}{u_1^2}=\frac{l^2}{1+s^2}\left\{1+\frac{e_1}{\sqrt{1+s^2}}\cos(v-\varpi_1)\right\}^{-2},$$

which can be expanded in powers of $e_1(1+s^2)^{-\frac{1}{2}}$ and cosines of multiples of $v-\varpi_1$. Expanding next the various powers of $\sqrt{1+s^2}$ in powers of s^2 and substituting for s its value, we shall obtain dt/dv expressed by means of cosines of multiples of $v-\varpi_1$ and $2(v-\theta)$. An integration will then give t in terms of v.

It will be noticed that ϖ_1, e_1 differ respectively from ϖ, e by quantities of the order γ^2.

53. The expansions given in Art. 50 will apply equally to the motion of the Sun, but become simpler since we suppose the plane of its orbit to be the plane of reference. Only four constants will be required; these will be called a', e', ϵ', ϖ'. The mean motion n' is defined properly by the equation

$$m'+\mu=m'+E+M=n'^2a'^3.$$

We have, in Arts. 19, 22, put $m'=n'^2a'^3$. The ratio $\mu:m'$ is approximately 1 : 330,000, so that the error caused will be very small.

We should properly have put in the disturbing function,

$$m'=n'^2a'^3-n^2a^3=n'^2a'^3(1-n^2a^3/n'^2a'^3).$$

The slight correction necessary is therefore obtained very easily after all the expansions, giving the motion of the Moon as disturbed by the Sun, have been made. The correction to be made to the largest coefficient in the expression of the longitude will not be so great as 0″·02.

It is to be remembered that a', n', e', ϵ', ϖ' are the constants of the ellipse which the Sun describes about the Centre of Mass of the Earth and Moon in accordance with the principles laid down in Chapter I.

54. An important question in connection with the series given in this Chapter is their convergence. Each coefficient of $\sin iw$ or $\cos iw$ is represented by a convergent series, and the series of coefficients thus arranged forms a convergent series as long as e is less than unity. But if we arrange the series according to powers of the eccentricity, this is no longer necessarily the case. We get in fact a double series, proceeding according to powers of e and sines or cosines of multiples of w, the convergency of which, for a certain range of values of e and w, depends on the manner of its arrangement. The problem is to find the greatest value of e for which the series is absolutely convergent. Laplace* has shown that if e be less than 0·6627432..., the series which have been discussed, together with those of the form $r^p \cos qf$, $r^p \sin qf$, will be absolutely convergent for all values of w. Full references will be found in Cayley's Report already referred to, and in Tisserand, Vol. I., Chap. XVI. In the latter are given some of the more important theorems on the subject.

* *Mém. de l'Inst. de France*, Vol. VI. (1823), pp. 61—80.

CHAPTER IV.

FORM OF SOLUTION. THE FIRST APPROXIMATION.

55. It has already been pointed out in Chapter II. that there is no known method of obtaining directly a general solution of the differential equations which express the motion of the Moon as disturbed by the Sun. In consequence, we are obliged to resort to indirect methods. There are two well-recognized devices used, both of which depend on the right to neglect certain parts of the equations of motion in the first instance, so as to reduce them to forms which are capable of integration, either by means of known functions or by the use of series, the coefficients of which can be found according to a definite law.

The Form to be given to the expressions of the Coordinates.

56. In discussing these methods of obtaining a solution of the general equations, it is necessary to keep certain physical considerations in view. It is not sufficient to obtain mere expressions for the coordinates: they must be put into such a form that practical applications may be possible and sufficiently simple. Since infinite series will be used, this point becomes of the greatest importance.

Now all records, ancient and modern, containing any mention of lunar observations—whether made in a scientific way or not—go to prove that, for a long period of time, the Moon has been circulating round the Earth in an orbit which is confined between limits not very far removed from one another. From this fact we infer that the motion is of such a nature that, at any rate during a considerable interval, its deviations from some mean state of motion (which, to fix ideas, we may think of as circular) are never very great. We ought then to try and express its coordinates in terms of the time, in a form which will give the position after any interval of time, whether it be short or long.

In order to be able to do this conveniently, the deviations from some mean state of motion ought to be expressible by functions which oscillate between finite and not very distant limits. The most convenient functions of this nature are the real periodic functions* sine and cosine. Should the variable, which is generally taken to be the time, occur, for example, in the form of a real exponential in the expression of a coordinate, such a term would cause the coordinate to increase indefinitely with t, for either positive or negative values of t. Again, should a term of the form $t^2 \sin nt$ be present, the same result would follow, provided the term has any independent existence. It may happen that such a term is present as one of a series which, in some other method of proceeding, would only have appeared through the expansion of some periodic function—an expansion only permissible for small values of t. As it is desired to obtain expressions holding also for large values of t, one object to be sought after is to try and obtain a solution in which such terms are not present. If they should arise, they must, if possible, be eliminated by some alteration in the *form* of the solution. In the case of the Moon's motion as disturbed by the Sun only, when the latter is moving in an elliptic orbit, it will be seen that the coordinates can be expressed by periodic terms only. See Art. 69.

A discussion of the limits, upper and lower, of the Moon's radius vector is given by G. W. Hill, *Researches in the Lunar Theory*, *Amer. Journ. Math.*, Vol. I., pp. 5–26. See also H. Gyldén, *Traité analytique des Orbits absolus*, etc. Vol. I., Chap. I.

Intermediate Orbits.

57. It has been stated in Art. 55, that the first step usually taken towards the determination of the Moon's path, is a simplification of the equations of motion, made by neglecting certain portions of them, to forms which can be readily integrated. A solution of the equations, thus limited, should form an approximation to the true path of the Moon if it is to be of assistance in obtaining the general solution of the complete equations of motion. This approximate path is called the *Intermediate*† *Orbit*, or more shortly the *Intermediary*. The intermediary need not necessarily be a general solution of the limited equations of motion; it may not contain the full number of arbitrary constants, but should be such that, when the general solution of the complete equations is required, it forms in some way an approximation to the path described by the Moon. It is not even necessary that the intermediary shall exactly satisfy the limited equations of motion. We may, after having obtained the exact solution of the latter, modify it in any way which experi-

* A periodic function is one which, after the addition of a definite and finite quantity to the variable, returns to its previous value. We consider periodic functions of a real variable only.

† This term was introduced by Gyldén. (German, intermediäre; Fr., intermédiaire.)

ence may suggest, provided that the modified solution differs from the exact solution by quantities of an order not less than the lowest order of the portions neglected in the original equations. The intermediary may then be indefinite to a certain extent, until we have found the general solution of the complete equations. Such indefiniteness will, however, be only allowed for the purpose of facilitating the analysis and in order to put the expressions for the coordinates into a suitable form, in accordance with the remarks made in Art. 56.

58. The next consideration is the choice of an intermediary. In this matter there is some freedom; it must partly depend on the particular method we intend to follow for the solution of the general equations. The usual plan is to choose an intermediate orbit which is such that, after a certain finite interval of time, the coordinates of any point on it, referred to axes which may be fixed or moving, return to the values they had at the beginning of the interval (an angular coordinate will have had its value increased by 2π). An orbit which possesses this property is called *periodic**. With reference to the axes used, the curve described will be closed.

59. In most of the older methods, the first limitation of the equations of motion is made by neglecting the action of the Sun, so that the intermediary is a fixed ellipse. A certain indefiniteness is then given by supposing the apsidal line and the nodal line of its orbit on a fixed plane to be moving with uniform angular velocities; these motions are determined on proceeding to the second and higher approximations. In other words, the intermediary is periodic with respect to moving axes. By this choice we begin by considering the problem of two bodies rather than the problem of three bodies.

Dr Hill starts from a different standpoint. He begins by neglecting, in the equations of motion, certain parts but not the whole, of the Sun's action, and he is able to obtain for the intermediary a solution, periodic with reference to axes moving in a definite manner; this is really a particular case of the problem of three bodies and the advantage of the orbit as a first approximation arises from this fact. This intermediary is not indeed a general solution of his limited equations (which were given in Art. 23), in that it does not possess the full number of arbitrary constants; nevertheless it serves as a useful first approximation owing to the fact that one of the arbitrary constants (the so-called 'eccentricity' of the Moon's orbit), which has been tacitly put equal to zero to get the intermediary, appears to be small enough to permit of expansions in ascending power series.

* On periodic solutions, see Poincaré, *Méc. Cél.* Vol. I. Chap. IV.

60. The subject of intermediate orbits has been treated by Gyldén, Andoyer, Hill and others. The usual plan is to express the disturbing function by powers of the ratio of the distances of the Sun and the Moon and by cosines of multiples of their angular distances; the coefficients and the term independent of this angle are then functions of the radii vectores of the Sun and Moon (or, in the planetary theory, of the two planets) only. All the terms containing this angle are neglected, so that the disturbing function involves only the radii vectores. A further simplification can be introduced by supposing the motion of the disturbing body to be circular. The case treated in Art. 67 is a simple illustration of the method followed. Gyldén* uses a method akin to Hansen's to solve the resulting equations; Andoyer† follows Laplace in taking the true longitude as independent variable; Hill‡ uses a direct method, by finding equations for the small differences $r-a$, $r'-a'$ (where a, a' are constants) and expanding in powers of them. The principal part of the motion of the perigee is determined without much difficulty.

When the intermediary has been obtained, there are two methods of proceeding to the solution of the general equations: (i) by continued approximation, (ii) by allowing the arbitrary constants introduced into the intermediary to vary.

(i) *Solution by continued Approximation.*

61. We suppose that, by means of the intermediary, the four variables, namely, the three coordinates and the time, have been expressed in terms of one of them; in these expressions there will be a certain number of the necessary six arbitrary constants present. With this solution or with a modified form of it, we then proceed to find what small corrections must be made to the variables when we include the omitted portions of the equations of motion. If the motion be stable, these corrections should take the form of small periodic terms. The method is then nothing else than that of small oscillations about a state of steady motion—that in the intermediate orbit. In the case of the Moon we shall generally have to proceed to the third and higher approximations in order to obtain the oscillations with sufficient accuracy. It is necessary to consider the amplitude, the period and the phase of each term.

(ii) *Solution by the Variation of Arbitrary Constants.*

62. The method is sufficiently well-known not to need explanation here§. In the case of the Moon we have three differential equations of the second order and therefore six arbitrary constants in the solution. We assume that an intermediate orbit has been found and that the resulting relations

* "Die intermediäres Bahn des Mondes," *Acta Math.*, Vol. VII. pp. 125–172 (1885).

† "Contribution à la Théorie des orbites intermédiaires," *Annales de la Fac. des Sc. de Toulouse*, Vol. I. M., pp. 1–72 (1887).

‡ "On Intermediate Orbits," *Annals of Math.* (U. S. A.), Vol. VIII. pp. 1–20 (1893).

§ See A. R. Forsyth, *Differential Equations*, Chapter IV.

between the coordinates and the time contain all the six arbitraries; it is required to find what variable values the arbitraries must have in order that the same relations may satisfy the general equations of motion. *The coordinates expressed in terms of the arbitraries and the time will thus have the same form for the intermediate orbit and the true orbit.* There are three relations, which may be chosen at will, between the first and second differentials of the arbitraries. These are always taken such that the first differentials of the coordinates have the same form whether the arbitraries be constant or variable. Hence, *the velocities, when expressed in terms of the arbitraries and the time, have the same form whether the arbitraries be constant or variable.* This way of stating the relations enables us to change from one system of coordinates to another without trouble. The six arbitraries and any function of them not involving the coordinates, the velocities or the time, are named *elements.* It is usual to take the undisturbed ellipse as the intermediary. The method will be treated in the following chapter and an important extension will be given to the meaning of the term 'element.'

The Instantaneous Ellipse.

63. We assume that the intermediary is an ellipse obtained when the action of the Sun is neglected. It is evident that if at any instant during the Moon's actual motion, the disturbing forces were to suddenly cease to act and the Moon were to continue its motion from that point under the mutual action of the Moon and the Earth only, it would describe an ellipse. This orbit is called the *Instantaneous Ellipse.*

Now when a particle is describing an ellipse under the Newtonian Law, if we are given the coordinates and the velocities* at any point, one ellipse can be constructed which satisfies the given conditions, and its six elements can be expressed uniquely in terms of the given coordinates and velocities. Conversely, the coordinates and velocities of the point considered can be determined uniquely in terms of the six elements. But since the coordinates and velocities of this point on the Instantaneous Ellipse are the same as those in the actual orbit, and since in the actual orbit the coordinates and velocities, when expressed by means of the arbitraries and the time, have the same form whether the arbitraries be constant or variable, the Instantaneous Ellipse is the Intermediate Orbit at the time when, in the expressions for the arbitraries, we have given to t the value which corresponds to the Moon's position at that instant. Hence, the elements of the Instantaneous Ellipse at any time t_1 can be obtained, after the solution by the method of the Variation of Arbitrary Constants has been carried out, by giving to t the value t_1 in the expressions which determine the arbitraries in terms of the time.

* That is, the magnitude and direction of the velocity.

Application of the Solution by continued Approximation.

64. Let us now return to the first method and see how it is to be applied to the solution of equations (A), Chapter II. We may begin by neglecting their right-hand members, that is, the terms dependent on the action of the Sun. The equations so limited will give the intermediate orbit—an ellipse of period $2\pi/n$; and we have seen in Chapter III. that, in this case, the coordinates can be expressed by sums of periodic functions of the time* which are sines and cosines of multiples of angles of the form $nt+\alpha$, where α is a constant.

To obtain the second approximation, we substitute these values of the coordinates in the right-hand members. But the disturbing function also depends on the coordinates of the Sun, which is supposed to move in an elliptic orbit of period $2\pi/n'$, and these coordinates will be expressed by sines and cosines of multiples of angles of the form $n't+\alpha'$. Since the time cannot enter into the right-hand members except through the coordinates, all the portions which depend on the action of the Sun will be periodic functions of the time and the arguments will be all of the form $int+i'n't+A$ (i, i' integers, positive, negative or zero and A a constant depending on the integers i, i' and on the longitudes of perigee, node and epoch of the two orbits). The equations being integrated, we obtain for our second approximation new values of the coordinates which, when substituted in the right-hand sides of equations (A), will, after a new integration, furnish a third approximation, and so on. This process is repeated until the desired accuracy is obtained.

65. Let us consider the nature of the equations which we obtain for the determination of the second approximation. The two integrals $\int d'R$ and $\int dt\, \partial R/\partial v$ must first be treated. It will be seen in Chapter VI. Art. 116, that the two expressions under the integral sign can contain no constant term when elliptic values in terms of the time have been substituted for the coordinates; therefore, unless n, n' are in the ratio of two whole numbers, no term directly proportional to the time can be introduced by these integrals. *We assume that n, n' are incommensurable.*

Hence the right-hand sides of equations (A) consist entirely of periodic terms, whose arguments are of the form $int+i'n't+A$. When therefore the first of these equations has been prepared for the second approximation, we may write it

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2)-\frac{\mu}{r}+\frac{\mu}{a}=-\mu\Sigma B\cos(int+i'n't+A),$$

where B is the constant coefficient corresponding to the argument $int+i'n't+A$.

* An angular coordinate will be considered to be periodic, if its rate of increase with respect to the time can be expressed by periodic functions only.

Let the elliptic value of r be r_0, and put

$$\frac{1}{r} = \frac{1}{r_0} + \delta u.$$

Then δu is a small quantity of the order of the disturbing forces. Since r_0 satisfies the equation when the right-hand member is put zero, we obtain for the left-hand member, by substituting the above value of r and neglecting powers of δu above the first,

$$-\frac{d^2}{dt^2}(r_0^3 \delta u) - \mu \delta u.$$

For the purposes here, since the eccentricity e is a small quantity, we shall neglect the product $e\delta u$ and therefore put $r_0^3 \delta u = a^3 \delta u$. Dividing by a^3 and giving to μ its value $n^2 a^3$, the equation becomes

$$\frac{d^2}{dt^2}\delta u + n^2 \delta u = n^2 \Sigma B \cos(int + i'n't + A).$$

66. This is a linear differential equation of well-known form*. Its solution consists of two parts—the Complementary Function, containing two arbitrary constants, and the Particular Integral. The former may be considered to be included in the first approximation, which already contains a similar expression with the requisite number of arbitrary constants. We are only concerned here with the Particular Integral. The latter is given by

$$\delta u = \Sigma \frac{n^2 B}{n^2 - (in + i'n')^2} \cos(int + i'n't + A).$$

There are two classes of terms included under the sign of summation—(*a*) those in which n is different from $in + i'n'$, (*b*) those in which $n = in + i'n'$ for some values of i, i'.

Case (*a*) is simple. The resulting terms in δu are of the same period as those of R and they constitute forced vibrations of which the periods are the same as those of the disturbing forces.

Case (*b*). Since n, n' are supposed incommensurable, this equality can only hold when i' is zero and therefore when $i = 1$. The corresponding particular integral is then of the form

$$-\Sigma \frac{B}{2} nt \sin(nt + A).$$

Proceeding to the third and higher approximations, it is evident that terms involving t^2, t^3... in the coefficients will appear. As such forms are contrary to the assumption of stable motion when only a finite number of them are

* A. R. Forsyth, *Differential Equations*, Chapter III. This particular form is given on pp. 61, 62.

taken, the question arises as to whether all these powers of t are not in reality the expansion of some periodic function—an expansion which cannot be convergent unless t be small—and whether it is not possible, by including certain portions of the Sun's action, to get a solution which shall consist of periodic terms only.

Modification of the Intermediate Orbit.

67. For this purpose we shall examine more closely the first of equations (A) and see how terms of period $2\pi/n$ may arise through the Sun's action. Neglect s the tangent of the latitude of the Moon, that is, suppose the motion to be in one plane; neglect also the ratio of the distances of the Moon and Sun. Putting $m' = n'^2a'^3$, the value of R given in Art. 7 will become

$$R = F - \frac{E+M}{r} = \left(\frac{a'}{r'}\right)^3 n'^2r^2\,[\tfrac{3}{2}\cos^2(v-v') - \tfrac{1}{2}],$$

where, as before, v, v' are the true longitudes of the Sun and the Moon. As we substitute elliptic values in the first approximation, we may still further limit the expression by neglecting e'—the solar eccentricity. Then

$$r' = a', \quad v' = n't + \epsilon'$$

and we have
$$R = n'^2r^2\,[\tfrac{1}{4} + \tfrac{3}{4}\cos 2\,(v - n't - \epsilon')].$$

Also, since r, v do not contain the angle $n't+\epsilon'$, when we substitute their elliptic values the second term of this expression will give no portion free from the angle $2n't+2\epsilon'$. As only those terms which produce arguments of the form $nt+A$ are sought, we limit R to its first term. Hence

$$R = \tfrac{1}{4}n'^2r^2, \quad \int d'R = \tfrac{1}{4}n'^2r^2 + \text{const.}, \quad r\frac{\partial R}{\partial r} = \tfrac{1}{2}n'^2r^2, \quad \frac{\partial R}{\partial v} = 0\,;$$

the equations (A) and (6) of Chap. II. now become

$$\tfrac{1}{2}\frac{d^2}{dt^2}(r^2) - \frac{\mu}{r} + \frac{\mu}{a} = n'^2r^2 + \text{const.}, \qquad \dot{v} = \frac{h}{r^2}, \qquad -\dot{v}^2 + \frac{\mu}{r^3} = \tfrac{1}{2}n'^2.$$

When the second and higher powers of n'^2 are neglected, the second and third of these equations give the steady motion

$$r = a\left(1 - \tfrac{1}{6}\frac{n'^2}{n^2}\right), \quad v = nt + \epsilon, \text{ if } n = \frac{h}{a^2}\left(1 + \tfrac{1}{3}\frac{n'^2}{n^2}\right), \quad \mu = n^2a^3:$$

by a suitable determination of the arbitrary constant in the first of the three equations of motion, this value of r will also satisfy it. It is required to find the small oscillations about this motion.

Let $r = a\left(1 - \tfrac{1}{6}\dfrac{n'^2}{n^2} + x\right)$. Neglecting powers and products of n'^2/n^2, x

beyond the first, we obtain from the substitution of this value in the equation for r,

$$\ddot{x} + (n^2 - \tfrac{3}{2}n'^2)\,x = 0.$$

The solution of this is given by

$$x = C\cos(cnt + D),$$

where $c^2n^2 = n^2 - \frac{3}{2}n'^2$, and therefore

$$c = 1 - \tfrac{3}{4}\frac{n'^2}{n^2}.$$

Hence the period of the oscillation differs from that of the original motion by a small quantity of the same order as the small term introduced.

If in finding the oscillation about the state of circular motion we had neglected n', the solution would have been

$$x = C\cos(nt + D),$$

which is nothing else than the second term of the elliptic expansion for r in powers of the eccentricity. If we expand the previous value of x in powers of n'^2/n^2, we get

$$x = C\cos(nt + D) + \tfrac{3}{4}(n'^2/n^2)\,ntC\sin(nt + D),$$

an expression which immediately shows how the occurrence of t in a coefficient took place.

68. In order then to make the equations (22) of Art. 50 available as a suitable first approximation we shall, in the terms dependent on the eccentricity, put $w = cnt + \epsilon - \varpi$, where c is a definite constant which differs from unity by quantities of the order of the disturbing forces and which is to be determined in the process of finding the second and higher approximations.

Exactly the same difficulty occurs in the equation for s, which will evidently give a form similar to that for δu when we proceed to a second approximation. The same artifice will serve. We put η instead of η_0 in the expressions, where $\eta = gnt + \epsilon - \theta$, g being a constant of the same nature as c.

The term $nt + \epsilon$ in v requires no modification since the difficulty does not arise in the longitude equation.

Hence, *the assumed first approximation to the solution of equations* (A) *of Chapter II. will be obtained by giving to v, r, s the values* (22) *of Art.* 50, *after we have substituted ϕ, η for w, η_0 respectively, where*

$$\phi = cnt + \epsilon - \varpi, \quad \eta = gnt + \epsilon - \theta.$$

It is evident that the same change would have been effected if we had substituted $(1 - c)\,nt + \varpi$ for ϖ and $(1 - g)\,nt + \theta$ for θ. A physical

meaning can therefore be given to these substitutions. Since ϖ and θ are the longitudes of the apse and node, the action of the Sun not only produces periodic oscillations about elliptic motion but also causes the apse and node to revolve. (Fuller explanations of the physical interpretation will be given in Chapter VIII.) The intermediary chosen may therefore be considered to be referred to moving axes.

For the subject of oscillations about a state of steady motion, E. J. Routh, *Rigid Dynamics*, Vol. II. Chap. VII. may be consulted; in particular, see Arts. 355–363 of the same Chapter.

69. Although by this modification of the intermediary we have succeeded in avoiding the occurrence of secular terms, there is no security that the expressions for the coordinates, consisting as they do of sums of periodic terms, will actually represent the values of the coordinates at any time. The periodic terms are infinite in number and, in order that they may give the true values of the coordinates at any time, they must form converging series for any value of t. At the present time little is known concerning the convergency of these series. In Poincaré's *Mécanique Céleste*, certain groups of the terms are shown to converge for sufficiently small values of the quantity in powers of which expansion is made, but no definite numerical results have been obtained except in the case of purely elliptic motion (Art. 54).

The comparison of theory with observation seems to indicate that we are justified in assuming that these series will represent the motion. Nevertheless it must be stated that Poincaré's investigations just referred to, show that a limited number of terms of a divergent series may, under particular circumstances, give with great accuracy the numerical values of the function which the series was intended to represent.

70. The remarks of the previous articles apply also to equations (11) of Chapter II. in which v is the independent variable. Before we can proceed to a second approximation it is necessary to express the coordinates of the Sun, which are given in terms of t, in terms of v; this causes no difficulty since we have found the elliptic value of t in terms of v in Chapter III. When this has been done, we substitute the elliptic values of u_1, s (Art. 52) in the terms depending on the action of the Sun. Putting $u_1=(u_1)_0+\delta u_1$, $s=s_0+\delta s$, the equations for δu_1, δs immediately take the linear form obtained for δu in Art. 65, with the difference that v is now the independent variable. A device similar to that used for equations (A) can be employed to avoid the presence of v in the coefficients of the periodic terms. We substitute in the first approximation $cv-\varpi$ for $v-\varpi$, $gv-\theta$ for $v-\theta$. It will be seen in Chap. VIII. that the constants c, g so defined are the same as those introduced in Art. 68.

CHAPTER V.

VARIATION OF ARBITRARY CONSTANTS.

71. THERE are several ways of applying the method of the variation of arbitrary constants (as outlined in Arts. 62, 63) to the problem of disturbed motion. The assumption that the coordinates and velocities, when expressed in terms of the arbitraries and the time, have the same form in the disturbed and undisturbed orbits, lies at the basis of all these investigations. The intermediate orbit is, in all cases, an ellipse obtained by neglecting the action of the Sun, and the six elements of this ellipse—or functions of them—are the arbitraries used.

The chapter is divided into two parts. The first part contains an elementary investigation of the differential equations which express the arbitraries in terms of the time when the action of the Sun is taken into account. In the second part, the equations for elliptic motion and for the arbitraries in disturbed motion are treated by the more powerful method of Jacobi. Certain results which will be required in later chapters follow.

(i) **Elementary ethods.**

72. We suppose that the equations for elliptic motion have been solved and that the coordinates and the velocities have been expressed in terms of the elements and of the time by means of the formulæ given in Chapter III. After proving certain preliminary propositions, the equations which give the variations of the six elements $a, e, \varpi, \epsilon, \theta, i$ in terms of the resolved parts of the disturbing forces in three directions, will be obtained. These equations will be then expressed in terms of the partial differential coefficients of R with respect to the elements. Finally we shall deduce the so-called 'canonical' system of equations used by Delaunay.

To find the change of position due to small arbitrary variations given to the elements.*

73. Consider a set of moving axes defined in fig. 5 by the points where they cut the unit sphere, the axis of Y being along the radius vector, the axis of X being $90°$ behind that of Y in the plane of the orbit and the axis of Z being perpendicular to this plane. Since the coordinates are supposed to be expressed in terms of the time and of the elements (Chap. III.), small changes in the latter will produce a change in the position of the Moon which may be defined by δr and by small rotations $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ of the axes of X, Y, Z about themselves. The point Y coincides with the point M of fig. 4, Art. 44.

Let zZ meet yx in C and let (as in Chap. III.) Ω be the node, $x\Omega=\theta$,

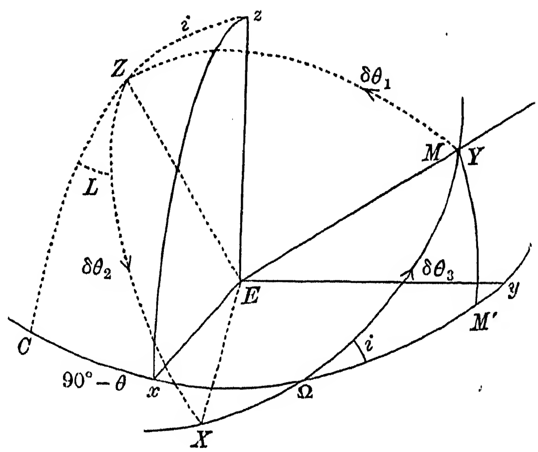

Fig. 5.

$zZ=y\Omega Y=i$, $CZX=\Omega Y=L$; hence $Cx=-xC=-(90-\theta)$. By Euler's geometrical equations†, we have then

$$\left.\begin{aligned}\delta\theta_1&=\sin L\delta i-\sin i\cos L\delta\theta\\ \delta\theta_2&=\cos L\delta i+\sin i\sin L\delta\theta\\ \delta\theta_3&=\delta\theta\cos i+\delta L\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(1).$$

Recurring to the notations of Art. 32, let δf, δE, δw, expressed in terms of the elements and of the time, denote the changes in f, E, w due to the variations δa, δe, $\delta\varpi$, $\delta\epsilon$, δn. These last are not all independent, owing to the equation $a^3n^2=\mu$. But since n, ϵ only occur in the coordinates in the form $nt+\epsilon$ (Art. 50), we can replace δn, $\delta\epsilon$ by the single variation $\delta\epsilon_1=t\delta n+\delta\epsilon$; the four variations δa, δe, $\delta\varpi$, $\delta\epsilon_1$ are then independent.

* The assumption laid down in Art. 71 is not introduced until Art. 77.

† E. J. Routh, *Rigid Dynamics*, Vol. I. Art. 256.

74. We have then the six arbitrary variations δa, δe, $\delta\varpi$, $\delta\epsilon_1$, $\delta\theta$, δi and these will produce changes $-r\delta\theta_3$, δr, $r\delta\theta_1$ in the position of the Moon (whose coordinates referred to the axes of X, Y, Z are $0, r, 0$) towards the positive directions of the axes. It is required to express the latter variations in terms of the former.

From Art. 32, we have $r = a(1 - e\cos E)$, $w = E - e\sin E$. Hence

$$\delta r = \frac{r}{a}\delta a - (a\cos E)\,\delta e + (ae\sin E)\,\delta E,$$

$$\delta E = \frac{a}{r}\delta w + \left(\frac{a}{r}\sin E\right)\delta e.$$

Therefore

$$\delta r = \frac{r}{a}\delta a + \frac{a^2 e\sin E}{r}\delta w + a\left(-\cos E + \frac{a}{r}e\sin^2 E\right)\delta e$$

$$= \frac{r}{a}\delta a + \frac{ae\sin f}{\sqrt{1-e^2}}\delta w + a\frac{-\cos E + e}{1 - e\cos E}\delta e,$$

by equations (2), (1) of Art. 32. But $w = nt + \epsilon - \varpi$ and therefore

$$\delta w = t\delta n + \delta\epsilon - \delta\varpi = \delta\epsilon_1 - \delta\varpi.$$

Hence, transforming the coefficient of δe by means of the relations of Art. 32,

$$\delta r = \frac{r}{a}\delta a + \frac{ae\sin f}{\sqrt{1-e^2}}(\delta\epsilon_1 - \delta\varpi) - (a\cos f)\,\delta e \quad\ldots\ldots\ldots\ldots(2).$$

Again $L = \Omega M =$ arg. of lat. $= f + \varpi - \theta$; then $\delta L = \delta f + \delta\varpi - \delta\theta$. We have (equation (3), Art. 32)

$$\frac{\delta f}{\sin f} = \frac{\delta e}{1-e^2} + \frac{\delta E}{\sin E} = \left(\frac{1}{1-e^2} + \frac{a}{r}\right)\delta e + \frac{a}{r\sin E}\delta w,$$

after the substitution of the value of δE given above. Hence, since

$$\sin f/\sin E = a\sqrt{1-e^2}/r,$$

we obtain

$$\delta L = \delta f + \delta\varpi - \delta\theta = \delta\varpi - \delta\theta + \sin f\left(\frac{1}{1-e^2} + \frac{a}{r}\right)\delta e + \frac{a^2}{r^2}\sqrt{1-e^2}\,(\delta\epsilon_1 - \delta\varpi);$$

and therefore, from (1),

$$\delta\theta_3 = \sin f\left(\frac{1}{1-e^2} + \frac{a}{r}\right)\delta e + \frac{a^2}{r^2}\sqrt{1-e^2}\,\delta\epsilon_1 + \left(1 - \frac{a^2}{r^2}\sqrt{1-e^2}\right)\delta\varpi - (1 - \cos i)\,\delta\theta \quad\ldots\ldots\ldots\ldots(3).$$

Since L is immediately expressible in terms of f and of the elements, the rotation $\delta\theta_1$ is immediately given by equations (1); $\delta\theta_2$ will not be required.

75. *To express the partial differential coefficients of R with respect to the elements in terms of the disturbing forces.*

We suppose that the values of the coordinates of the Moon, given in Art. 50, have been substituted in R; R will then be a function of a, e, $nt+\epsilon$, ϖ, θ, i and of the coordinates of the Sun: the latter, being expressed by elliptic formulæ, are considered known functions of the time and of definite constants and they can therefore be left out of consideration.

The changes in the elements of the Moon denoted by the symbol δ, have produced changes $-r\delta\theta_3$, δr, $r\delta\theta_1$ in the position, towards the positive directions of the axes of X, Y, Z. Let the disturbing forces in these three directions be $-\mathfrak{T}$, $\mathfrak{P}$, $\mathfrak{Z}$. Then $\mathfrak{P}$ acts along the radius vector, $\mathfrak{T}$ perpendicular to it in the direction of motion and $\mathfrak{Z}$ perpendicular to the plane of the orbit.

The Virtual Work done by the forces is

$$-\mathfrak{T}(-r\delta\theta_3)+\mathfrak{P}\delta r+\mathfrak{Z}r\delta\theta_1.$$

Let the corresponding change in R be δR. Since the change in position is produced by variations of the elements only, the Virtual Work is δR and

$$\mathfrak{P}\delta r+\mathfrak{T}r\delta\theta_3+\mathfrak{Z}r\delta\theta_1=\delta R$$

$$=\frac{\partial R}{\partial a}\delta a+\frac{\partial R}{\partial e}\delta e+\frac{\partial R}{\partial(nt+\epsilon)}\delta\epsilon_1+\frac{\partial R}{\partial\varpi}\delta\varpi+\frac{\partial R}{\partial\theta}\delta\theta+\frac{\partial R}{\partial i}\delta i.$$

Substitute in this equation the values of δr, $\delta\theta_3$, $\delta\theta_1$ previously obtained; since the variations of a, e, $nt+\epsilon$, ϖ, θ, i are independent, we can equate to zero their coefficients. The six resulting equations will give the values of $\frac{\partial R}{\partial a}\cdots\frac{\partial R}{\partial i}$ in terms of $\mathfrak{P}$, $\mathfrak{T}$, $\mathfrak{Z}$. Before writing them down we notice that since ϵ never occurs except in the form $nt+\epsilon$,

$$\frac{\partial R}{\partial(nt+\epsilon)}=\frac{\partial R}{\partial\epsilon}.$$

Also, $\partial R/\partial a$ is taken with reference to a, only as a occurs explicitly and not as it occurs through n.

The resulting equations are easily found to be

$$\left.\begin{aligned}
\frac{\partial R}{\partial a}&=\mathfrak{P}\frac{r}{a},\\
\frac{\partial R}{\partial e}&=-\mathfrak{P}a\cos f+\mathfrak{T}a\left(\frac{r}{a(1-e^2)}+1\right)\sin f,\\
\frac{\partial R}{\partial\epsilon}&=\mathfrak{P}\frac{ae}{\sqrt{1-e^2}}\sin f+\mathfrak{T}\frac{a^2}{r}\sqrt{1-e^2},\\
\frac{\partial R}{\partial\varpi}&=-\mathfrak{P}\frac{ae}{\sqrt{1-e^2}}\sin f-\mathfrak{T}\frac{a^2}{r}\sqrt{1-e^2}+\mathfrak{T}r,\\
\frac{\partial R}{\partial\theta}&=-2\mathfrak{T}r\sin^2\tfrac{1}{2}i-\mathfrak{Z}r\sin i\cos L,\\
\frac{\partial R}{\partial i}&=\mathfrak{Z}r\sin L
\end{aligned}\right\}\quad\text{............(4).}$$

Corollary. We deduce immediately

$$\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon} = \mathfrak{T}r,$$

$$\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \theta} = \mathfrak{T}r \cos i - \mathfrak{Z}r \sin i \cos L.$$

If r_1 be the projected radius vector and v the longitude in the fixed plane, $\partial R/r_1 \partial v$ is the disturbing force perpendicular to the projection of the radius vector in the fixed plane. Hence, resolving in this direction, we have by fig. 5, if MM' be perpendicular to xy,

$$\begin{aligned} \frac{\partial R}{\partial v} &= r_1 (\mathfrak{T} \sin \Omega MM' - \mathfrak{Z} \cos \Omega MM') \\ &= r (\mathfrak{T} \cos MM' \sin \Omega MM' - \mathfrak{Z} \cos MM' \cos \Omega MM') \\ &= r (\mathfrak{T} \cos i - \mathfrak{Z} \sin i \cos L), \end{aligned}$$

whence
$$\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \theta} = \frac{\partial R}{\partial v}.$$

The expression $\partial R/\partial v$ implicitly supposes that R is expressed in terms of r_1, v, z. (See Art. 13.)

76. Let ϕ be any function of the elements and of the time. The symbol $\delta\phi$ denotes the change in ϕ arising from the changes in the elements only and therefore $\delta\phi/dt$ denotes differentiation of ϕ with respect to t, only in so far as t occurs through the variability of the elements and not through its presence explicitly in ϕ. If ϕ is a function of the elements only,

$$\delta\phi/dt = d\phi/dt.$$

Thus
$$\frac{\delta(nt+\epsilon)}{dt} = t\frac{\delta n}{dt} + \frac{\delta\epsilon}{dt} = t\frac{dn}{dt} + \frac{d\epsilon}{dt}.$$

Also, we denote by $\partial\phi/\partial t$ the differential coefficient of ϕ with respect to t, only in so far as t occurs *explicitly* in ϕ. Then

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\delta\phi}{dt}.$$

As $\partial r/\partial t$ occurs frequently in the following articles we shall denote it by $\dot{r}$.

To find the differential equations required in order to express the elements in terms of the time in disturbed motion.

77. According to the principles laid down for forming these equations, the coordinates and velocities, when expressed in terms of the elements and of the time, are to have the same form whether the motion be undisturbed or

disturbed. Hence *the part of the change in position, due to the variability of the elements alone, is zero.*

Let the variations $\delta a, \ldots \delta i$ of the elements be now the changes which *actually* take place in time dt, owing to the disturbing forces (see Art. 91). Then $-r\delta\theta_3$, δr, $r\delta\theta_1$, become the changes in position in time dt, due to the variability of the elements only. We therefore have

$$\frac{\delta r}{dt}=0, \quad r\frac{\delta\theta_3}{dt}=0, \quad r\frac{\delta\theta_1}{dt}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Similarly $$\delta v/dt=0.$$

The three equations of motion of the Moon may be replaced by

$$\left.\begin{aligned}&\frac{d^2r}{dt^2}-r\frac{d\theta_3^2}{dt^2}=-\frac{\mu}{r^2}+\mathfrak{P},\\ &\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta_3}{dt}\right)=\mathfrak{T},\\ &\frac{1}{r_1}\frac{d}{dt}\left(r_1^2\frac{dv}{dt}\right)=\frac{1}{r_1}\frac{\partial R}{\partial v}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

For, by definition, $d\theta_3$ is the angle, reckoned in the plane of the orbit, between two consecutive positions of the radius vector; instead of the equation for the motion perpendicular to the plane of the orbit, we use the third of the above equations which, by the Corollary to Art. 75, introduces the force $\mathfrak{Z}$. The first two equations may also be deduced from the general formulæ* for the motion of a point whose coordinates are 0, r, 0, referred to the moving axes used here, by putting $\delta\theta_1=0=d\theta_1$.

When the motion is undisturbed, we have

$$\left.\begin{aligned}&\frac{\partial^2 r}{\partial t^2}-r\frac{\partial\theta_3^2}{\partial t^2}=-\frac{\mu}{r^2},\\ &\frac{1}{r}\frac{\partial}{\partial t}\left(r^2\frac{\partial\theta_3}{\partial t}\right)=0,\\ &\frac{1}{r_1}\frac{\partial}{\partial t}\left(r_1^2\frac{\partial v}{\partial t}\right)=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

But since $d\theta_3/dt=\partial\theta_3/\partial t+\delta\theta_3/dt$ and since by (5) $\delta\theta_3/dt=0$, etc. we have

$$\frac{\partial\theta_3}{\partial t}=\frac{d\theta_3}{dt}, \quad \dot{r}=\frac{\partial r}{\partial t}=\frac{dr}{dt}, \quad \frac{\partial v}{\partial t}=\frac{dv}{dt}.$$

From the second of these we get

$$\frac{d^2r}{dt^2}=\frac{\partial^2 r}{\partial t^2}+\frac{\delta\dot{r}}{dt}.$$

* E. J. Routh, *Rigid Dynamics*, Vol. I. Art. 238.

Let $$r^2\frac{\partial\theta_3}{\partial t}=h_0=na\sqrt{1-e^2},\quad r_1{}^2\frac{\partial v}{\partial t}=h=h_0\cos i \text{ (Art. 51)}.$$

We have then, by the subtraction of equations (7) from (6),

$$\frac{\delta\dot{r}}{dt}=\mathfrak{P},\quad \frac{\delta h_0}{dt}=\mathfrak{T}r,\quad \frac{\delta h}{dt}=\frac{\partial R}{\partial v}\quad\text{......................(8).}$$

We shall deduce the required formulæ from (5), (8).

78. If we refer to fig. 5, we see that since $\delta\theta_1$, $\delta\theta_3$ are, by Art. 77, both zero, the moving axes, as far as their rotation is due to the disturbing forces only, have the single rotation $\delta\theta_2$; *the instantaneous axis is therefore the radius vector.* Hence, to get from one point to a consecutive point in the actual orbit when the values of the elements at the moment under consideration are given, we calculate the displacement in the plane of the orbit by means of the elliptic formulæ and then give the orbit a rotation $\delta\theta_2$ about the radius vector. The effect of this rotation on the position will be of the second order of small quantities.

To obtain the rate of rotation of the orbit, we have from equations (1),

$$\delta\theta_1{}^2+\delta\theta_2{}^2=\delta i^2+\sin^2 i\,\delta\theta^2.$$

Hence, since $\delta\theta_1=0$,

$$\frac{d\theta_2}{dt}=\sqrt{\frac{di^2}{dt^2}+\sin^2 i\frac{d\theta^2}{dt^2}}.$$

79. Any line in the plane of the orbit which has no rotation about the axis of Z, is said to be *fixed in the plane of the orbit.* Such a line will be absolutely fixed when the motion is undisturbed; when the motion is disturbed it will move only with the *plane* of the orbit. The point where such a line cuts the unit sphere is also fixed in the plane of the orbit and has been termed by Cayley* a *Departure Point.* When the plane of the orbit is in motion, the line joining any two consecutive positions of a departure point is perpendicular to the intersection of the orbit with the unit sphere. Hence the curves described by departure points cut the plane of the orbit at any time orthogonally.

Since the variation in the position of the radius vector, due to the disturbing forces only, is zero, longitudes and angular velocities reckoned in the plane of the orbit from the departure point have the same form whether the motion be disturbed or undisturbed.

80. We shall first find the equations for the variations of the elements in terms of the forces $\mathfrak{P}$, $\mathfrak{T}$, $\mathfrak{Z}$: these will be required in Chap. X. It will be then easy to deduce their values in terms of the partial differential coefficients of R with respect to the elements by means of equations (4).

The Inclination and the Longitude of the Node.

We have, from equations (8) since h_0 is a function of the elements only,

$$\frac{dh_0}{dt}=\mathfrak{T}r,\quad \frac{d(h_0\cos i)}{dt}=\frac{\partial R}{\partial v}.$$

* On Hansen's Lunar Theory. *Quart. Journ. Math.* Vol. I. pp. 112–125, *Coll. Works,* Vol. III. p. 19.

Hence $$\frac{\partial R}{\partial v} = \frac{dh_0}{dt}\cos i - h_0 \sin i \frac{di}{dt} = \mathfrak{T}r\cos i - h_0\sin i\frac{di}{dt}.$$

Substituting for $\partial R/\partial v$ in terms of $\mathfrak{T}$, $\mathfrak{Z}$ (Cor. Art. 75) we obtain, after division by $\sin i$,

$$\frac{di}{dt} = \mathfrak{Z}\frac{r}{h_0}\cos L \text{ ....} \quad (9).$$

Whence, from the first of equations (1), since $\delta\theta_1/dt = 0$,

$$\sin i\frac{d\theta}{dt} = \mathfrak{Z}\frac{r}{h_0}\sin L \text{ ....} \quad (10).$$

The Major Axis.

We have in the ellipse,

$$\dot{r}^2 + \frac{h_0^2}{r^2} = \frac{2\mu}{r} - \frac{\mu}{a}.$$

Submitting this to the operation δ/dt, we get, since $\delta\dot{r}/dt = \mathfrak{P}$, $\delta r/dt = 0$, $\delta h_0/dt = \mathfrak{T}r$, $\delta a/dt = da/dt$,

$$2\dot{r}\mathfrak{P} + \frac{2h_0}{r^2}\mathfrak{T}r = \frac{\mu}{a^2}\frac{da}{dt}.$$

Whence, inserting the value of $\dot{r}$ obtained by putting $dr/dw = \dot{r}/n$ in Art. 32,

$$\frac{da}{dt} = \frac{2na^3 e\sin f}{\mu\sqrt{1-e^2}}\mathfrak{P} + \frac{2h_0 a^2}{\mu r}\mathfrak{T} \text{........................} (11).$$

The Eccentricity.

We have $$h_0^2 = \mu a(1-e^2).$$

Differentiating and putting $\mathfrak{T}r$ for dh_0/dt, we obtain

$$\begin{aligned} 2h_0\mathfrak{T}r &= \mu(1-e^2)\frac{da}{dt} - 2\mu ae\frac{de}{dt} \\ &= 2na^3 e\sqrt{1-e^2}\,\mathfrak{P}\sin f + \frac{2h_0 a^2}{r}(1-e^2)\mathfrak{T} - 2\mu ae\frac{de}{dt}, \end{aligned}$$

by equation (11). Whence, since $na^2\sqrt{1-e^2} = h_0$,

$$\frac{de}{dt} = \frac{h_0}{\mu}\mathfrak{P}\sin f + \frac{h_0}{\mu e}\left\{\frac{a(1-e^2)}{r} - \frac{r}{a}\right\}\mathfrak{T} \text{..................} (12).$$

As $a(1-e^2)/r = 1 + e\cos f$, $r/a = 1 - e\cos E$, this may be also put into the form

$$\frac{de}{dt} = \frac{h_0}{\mu}\mathfrak{P}\sin f + \frac{h_0}{\mu}\mathfrak{T}(\cos f + \cos E) \text{..................} (12').$$

The Longitude of Perigee.

Since $h_0^2 = \mu l$, we have $\dot{r}/\sin f = \mu e/h_0$, and therefore

$$\dot{r}\cot f = \frac{\mu e \cos f}{h_0} = \frac{\mu}{h_0}\left(\frac{l}{r} - 1\right) = \frac{h_0}{r} - \frac{\mu}{h_0}.$$

Applying to this the operation δ/dt and substituting as before, we obtain

$$\mathfrak{P}\cot f - \frac{\dot{r}}{\sin^2 f}\frac{\delta f}{dt} = \left(\frac{1}{r} + \frac{\mu}{h_0^2}\right)\mathfrak{T}r.$$

Whence, since $h_0^2 = \mu l$, $na = \mu/na^2$, we have, after inserting the value of $\dot{r}$,

$$e\frac{\delta f}{dt} = \frac{na^2}{\mu}\sqrt{1-e^2}\,\mathfrak{P}\cos f - \frac{na^2}{\mu}\sqrt{1-e^2}\left(1 + \frac{r}{l}\right)\mathfrak{T}\sin f.$$

But since $\delta\theta_3/dt = 0$, the third of equations (1) gives

$$0 = \frac{d\theta}{dt}\cos i + \frac{\delta L}{dt} = \frac{d\theta}{dt}\cos i + \frac{\delta f}{dt} + \frac{d\varpi}{dt} - \frac{d\theta}{dt} \quad\text{..............(13).}$$

Therefore, substituting for $\delta f/dt$,

$$e\frac{d\varpi}{dt} = 2e\sin^2\tfrac{1}{2}i\frac{d\theta}{dt} + \frac{na^2\sqrt{1-e^2}}{\mu}\left\{-\mathfrak{P}\cos f + \mathfrak{T}\left(1 + \frac{r}{l}\right)\sin f\right\}\text{...(14).}$$

The Epoch.

We have, from the equations (4) and (2) of Art. 32,

$$\dot{r} = \frac{nae}{\sqrt{1-e^2}}\sin f = \frac{na^2 e}{r}\sin E,$$

and therefore, since $\mu = n^2a^3$,

$$\dot{r}^2 = \frac{\mu e}{r\sqrt{1-e^2}}\,e\sin E\sin f.$$

Taking logarithms and applying the operation δ/dt, we obtain

$$\frac{2\mathfrak{P}}{\dot{r}} = \frac{1}{e(1-e^2)}\frac{de}{dt} + \frac{1}{e\sin E}\frac{\delta(e\sin E)}{dt} + \cot f\frac{\delta f}{dt}.$$

But, from equations (3) of Art. 32, we deduce

$$\frac{\delta(e\sin E)}{dt} = \frac{\delta E}{dt} - \frac{\delta w}{dt}, \qquad \frac{1}{1-e^2}\frac{de}{dt} + \frac{1}{\sin E}\frac{\delta E}{dt} = \frac{1}{\sin f}\frac{\delta f}{dt}.$$

Substituting for $\delta(e\sin E)/dt$ and then for $\delta E/dt$ in the previous equation, we shall find that de/dt disappears and that the equation becomes

$$\frac{2\mathfrak{P}}{\dot{r}} + \frac{1}{e\sin E}\frac{\delta w}{dt} = \left(\frac{1}{e\sin f} + \frac{\cos f}{\sin f}\right)\frac{\delta f}{dt} = \frac{a(1-e^2)}{er\sin f}\frac{\delta f}{dt}.$$

Putting $\dot{r}r = na^2 e \sin E$, $r \sin f = a\sqrt{1-e^2} \sin E$ (Art. 32), this equation reduces to

$$\frac{2r}{na^2}\mathfrak{P} + \frac{\delta w}{dt} = \sqrt{1-e^2}\frac{\delta f}{dt}$$

$$= \sqrt{1-e^2}\left(2\sin^2 \tfrac{1}{2}i\frac{d\theta}{dt} - \frac{d\varpi}{dt}\right), \text{ by equation (13).}$$

Finally, since $\delta w/dt = d\epsilon_1/dt - d\varpi/dt$ (Art. 74), we obtain

$$\frac{d\epsilon_1}{dt} = 2\sqrt{1-e^2}\sin^2 \tfrac{1}{2}i\frac{d\theta}{dt} + (1-\sqrt{1-e^2})\frac{d\varpi}{dt} - \frac{2nar}{\mu}\mathfrak{P} \quad \text{...... (15),}$$

which, by the help of equations (10), (14), gives the value of $\delta\epsilon_1/dt$.

81. We might of course immediately deduce the value of $d\epsilon/dt$ from this by obtaining the value of dn/dt from that of da/dt in (11). But its value introduces the time in the form $t\,dn/dt$, which possesses the inconvenience mentioned in Art. 66.

From the definition of ϵ_1, we have

$$\epsilon_1 = \epsilon + \int t\,dn = \epsilon + nt - \int n\,dt,$$

or

$$nt + \epsilon = \int n\,dt + \epsilon_1.$$

So that by substituting $\int n\,dt$ for nt, we change ϵ into ϵ_1. Since n only occurs explicitly in the form $nt+\epsilon$, we shall consider the substitution to have been made. With this understanding the suffix of ϵ_1 is very generally omitted. The integral $\int n\,dt$ is called *the mean motion in the disturbed orbit*.

82. The results obtained may be written, after a few small changes:

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2na^2}{\mu}\left(\mathfrak{P}\frac{ae}{\sqrt{1-e^2}}\sin f + \mathfrak{T}\frac{a^2}{r}\sqrt{1-e^2}\right),\\
\frac{de}{dt} &= \frac{na(1-e^2)}{\mu e}\left\{\mathfrak{P}\frac{ae}{\sqrt{1-e^2}}\sin f + \mathfrak{T}\frac{a^2}{r}\sqrt{1-e^2}\right\} - \frac{na\sqrt{1-e^2}}{\mu e}\mathfrak{T}r,\\
\frac{d\varpi}{dt} &= \frac{na\sqrt{1-e^2}}{\mu e}\left\{-\mathfrak{P}a\cos f + \mathfrak{T}a\left(1+\frac{r}{a(1-e^2)}\right)\sin f\right\} + 2\sin^2\tfrac{1}{2}i\frac{d\theta}{dt},\\
\frac{d\epsilon_1}{dt} &= -\frac{2nar}{\mu}\mathfrak{P} + (1-\sqrt{1-e^2})\frac{d\varpi}{dt} + 2\sqrt{1-e^2}\sin^2\tfrac{1}{2}i\frac{d\theta}{dt},\\
\frac{d\theta}{dt} &= \frac{nar}{\mu\sqrt{1-e^2}}\mathfrak{Z}\frac{\sin L}{\sin i},\\
\frac{di}{dt} &= \frac{nar}{\mu\sqrt{1-e^2}}\mathfrak{Z}\cos L
\end{aligned}\right\} \quad \text{...(16).}$$

83. Finally, we desire to express the terms on the right hand of equations (16) by means of the partial differentials of R with respect to the

elements and, if possible, with coefficients which are functions of the elements *only*. This can be done by means of the equations (4). None of the substitutions present any difficulty. The results are as follows :—

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2na^2}{\mu}\frac{\partial R}{\partial \epsilon},\\
\frac{de}{dt} &= \frac{na(1-e^2)}{\mu e}\frac{\partial R}{\partial \epsilon} - \frac{na\sqrt{1-e^2}}{\mu e}\left(\frac{\partial R}{\partial \epsilon}+\frac{\partial R}{\partial \varpi}\right),\\
\frac{d\varpi}{dt} &= \frac{na\sqrt{1-e^2}}{\mu e}\frac{\partial R}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\tfrac{1}{2}i\,\frac{\partial R}{\partial i},\\
\frac{d\epsilon_1}{dt} &= -\frac{2na^2}{\mu}\frac{\partial R}{\partial a} + \frac{na\sqrt{1-e^2}}{\mu e}\left(1-\sqrt{1-e^2}\right)\frac{\partial R}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\tfrac{1}{2}i\,\frac{\partial R}{\partial i},\\
\frac{d\theta}{dt} &= \frac{na}{\mu\sqrt{1-e^2}}\frac{1}{\sin i}\frac{\partial R}{\partial i},\\
\frac{di}{dt} &= -\frac{na}{\mu\sqrt{1-e^2}}\left\{\frac{1}{\sin i}\frac{\partial R}{\partial \theta} + \tan\tfrac{1}{2}i\left(\frac{\partial R}{\partial \epsilon}+\frac{\partial R}{\partial \varpi}\right)\right\}
\end{aligned}\right\}\ldots(17).$$

These equations are obtained without the intervention of 𝔓, 𝔗, 𝔜 by C. H. H. Cheyne, *Planetary Theory*, Chap. II.

The ordinary Canonical System of Equations.

84. The system of equations just obtained is by no means the simplest in form; by taking certain functions of the elements used above, to form a new set of elements, we can reduce the equations to a very convenient form. Let

$$\left.\begin{aligned}
\alpha_1 &= (\epsilon-\varpi)/n, & \alpha_2 &= \varpi-\theta, & \alpha_3 &= \theta\\
\beta_1 &= -\mu/2a, & \beta_2 &= h_0, & \beta_3 &= h_0\cos i
\end{aligned}\right\}\ldots\ldots\ldots(18).$$

The equations for the new elements α, β take the form, known as *canonical*,

$$\left.\begin{aligned}
\frac{d\beta_1}{dt} &= \frac{\partial R}{\partial \alpha_1}, & \frac{d\beta_2}{dt} &= \frac{\partial R}{\partial \alpha_2}, & \frac{d\beta_3}{dt} &= \frac{\partial R}{\partial \alpha_3}\\
\frac{d\alpha_1}{dt} &= -\frac{\partial R}{\partial \beta_1}, & \frac{d\alpha_2}{dt} &= -\frac{\partial R}{\partial \beta_2}, & \frac{d\alpha_3}{dt} &= -\frac{\partial R}{\partial \beta_3}
\end{aligned}\right\}\ldots\ldots\ldots\ldots(19),$$

where R is now supposed expressed in terms of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, t$.

85. To prove these we first notice that since the second three equations in (18) do not contain ϵ, ϖ or θ, we can immediately deduce from the first three*

$$\frac{\partial R}{\partial \theta} = -\frac{\partial R}{\partial \alpha_2} + \frac{\partial R}{\partial \alpha_3}, \qquad \frac{\partial R}{\partial \varpi} = -\frac{1}{n}\frac{\partial R}{\partial \alpha_1} + \frac{\partial R}{\partial \alpha_2}, \qquad \frac{\partial R}{\partial \epsilon} = \frac{1}{n}\frac{\partial R}{\partial \alpha_1}.$$

* It is to be noticed that the expressions $\partial R/\partial\theta\ldots$ suppose that R is expressed in terms of $a, e\ldots i$ and $\partial R/\partial\alpha_1\ldots$ that R is expressed in terms of $\alpha_1, \alpha_2\ldots\beta_3$.

For, since θ is contained only in a_2, a_3, we have

$$\frac{\partial R}{\partial \theta}=\frac{\partial R}{\partial a_2}\frac{\partial a_2}{\partial \theta}+\frac{\partial R}{\partial a_3}\frac{\partial a_3}{\partial \theta}=-\frac{\partial R}{\partial a_2}+\frac{\partial R}{\partial a_3},$$

and so on. Hence, by the Cor., Art. 75,

$$\frac{\partial R}{\partial a_1}=n\frac{\partial R}{\partial \epsilon},\quad \frac{\partial R}{\partial a_2}=\frac{\partial R}{\partial \epsilon}+\frac{\partial R}{\partial \varpi}=\mathfrak{T}r,\quad \frac{\partial R}{\partial a_3}=\frac{\partial R}{\partial \epsilon}+\frac{\partial R}{\partial \varpi}+\frac{\partial R}{\partial \theta}=\frac{\partial R}{\partial v}.$$

We therefore have, by equations (17), (8),

$$\frac{d\beta_1}{dt}=\frac{\mu}{2a^2}\frac{da}{dt}=n\frac{\partial R}{\partial \epsilon}=\frac{\partial R}{\partial a_1},$$

$$\frac{d\beta_2}{dt}=\frac{dh_0}{dt}=\mathfrak{T}r=\frac{\partial R}{\partial a_2},$$

$$\frac{d\beta_3}{dt}=\frac{d\,(h_0\cos i)}{dt}=\frac{\partial R}{\partial v}=\frac{\partial R}{\partial a_3},$$

giving the first three of equations (19).

86. Again, by the fifth of equations (17), we have

$$\frac{da_3}{dt}=\frac{d\theta}{dt}=\frac{1}{h_0\sin i}\cdot\frac{\partial R}{\partial i}=-\frac{\partial R}{h_0\partial\cos i}=-\frac{\partial R}{\partial \beta_3},$$

for i only enters into R through the element β_3.

Also, by the third and fifth of the same equations,

$$\frac{da_2}{dt}=\frac{d\varpi}{dt}-\frac{d\theta}{dt}=\frac{h_0}{\mu e a}\frac{\partial R}{\partial e}-\frac{\cos i}{h_0\sin i}\frac{\partial R}{\partial i}$$

$$=\frac{h_0}{\mu e a}\frac{\partial R}{\partial e}+\cos i\,\frac{\partial R}{\partial \beta_3}.$$

But since e enters into R only through β_2, β_3, we have

$$\frac{\partial R}{\partial e}=\frac{\partial R}{\partial \beta_2}\frac{\partial \beta_2}{\partial e}+\frac{\partial R}{\partial \beta_3}\frac{\partial \beta_3}{\partial e}=-\left(\frac{\partial R}{\partial \beta_2}+\cos i\,\frac{\partial R}{\partial \beta_3}\right)\frac{na^2e}{\sqrt{1-e^2}},$$

since $h_0=na\sqrt{1-e^2}$. Substituting this value of $\partial R/\partial e$ in the previous equation we obtain, after putting $\mu=n^2a^3$, the equation $da_2/dt=-\partial R/\partial\beta_2$.

Finally, since $\delta r/dt$, $\delta\theta_1/dt$, $\delta\theta_3/dt$, are all zero and since the variations δ are now those which actually take place, we have from Art. 75,

$$\frac{\delta R}{dt}=0.$$

Therefore
$$\frac{\partial R}{\partial a_1}\frac{da_1}{dt}+\frac{\partial R}{\partial a_2}\frac{da_2}{dt}+\frac{\partial R}{\partial a_3}\frac{da_3}{dt}+\frac{\partial R}{\partial \beta_1}\frac{d\beta_1}{dt}+\frac{\partial R}{\partial \beta_2}\frac{d\beta_2}{dt}+\frac{\partial R}{\partial \beta_3}\frac{d\beta_3}{dt}=0.$$

Substituting for $\frac{da_2}{dt}$, $\frac{da_3}{dt}$, $\frac{d\beta_1}{dt}$, $\frac{d\beta_2}{dt}$, $\frac{d\beta_3}{dt}$ the values just found, the second and fifth and the third and sixth terms respectively cancel one another; after division by $\partial R/\partial a_1$ the equation becomes $da_1/dt=-\partial R/\partial\beta_1$.

87. All systems of elements which satisfy equations of the form (19) are said to be *canonical.*

Other canonical systems of elements and the conditions which must be satisfied in order to transform from one canonical system to another, will be found in the works of Jacobi, Dziobek and Poincaré referred to below. The general form of this transformation is that known as tangential (Berührungstransformation).

88. The method of treatment given in this Chapter—that of causing the elements to vary in order to include the disturbing forces—is more generally useful in its applications to the planetary than to the lunar theory. The equations for the variations do not admit, any more than the equations of motion examined in Chap. II., of a direct solution and we are obliged here also to use some method of approximation. This proceeds according to the plan explained in Chap. IV. We first find the values of $\partial R/\partial a$... so that the right-hand sides of the equations (17) become functions of the time and of the elements. To solve, in general we may first consider the elements on the right-hand side to be constant—or we may combine the equations in any suitable manner to make them integrable; we thus obtain the values of the elements in terms of the time and of six new arbitraries. Using these new values in the terms on the right-hand sides, we again get the latter expressed as functions of the time and of absolute constants and we can proceed in this way until the desired accuracy is obtained; the new arbitraries introduced at each step can be determined so as to simplify the final expressions as much as possible.

In the lunar theory, the necessity for a large number of terms and for many approximations causes the process to become very tedious. Delaunay's theory (Chap. IX.) —the only one worked out on these lines—is very fully expanded, but the labour of obtaining the expressions was enormous and the results leave much to be desired. It is also to be remembered that we cannot start by giving the constants their numerical values—a literal development is usually essential. Hansen's theory (Chap. X.) is not really treated after this method. He uses the variable arbitrary constants in order to obtain certain functions for the motion in the instantaneous plane but, having done so, he is able to use numerical values for his constants from the outset.

In the planetary theory, secular terms—that is, terms increasing in proportion with the time—appear, and also terms with large coefficients and of long period: these are very much more easily managed by considering them as attached to the elements than by considering them as corrections to the coordinates.

89. One of the most important properties of the equations—and of the corresponding equations for all sets of elements which may be used—is the fact that the coefficients of the partials $\partial R/\partial \lambda$ (where λ is any element) are independent of the time explicitly, that is, they are functions of the elements alone. The time only occurs explicitly on the right-hand sides through the presence of the coordinates of the Sun in R. See Art. 99.

It will be noticed that the method practically replaces three differential equations of the second order by six of the first order. For obtaining literal developments of the coordinates this is of doubtful advantage, but for theoretical investigations it is of the highest importance. Canonical systems of elements, as used by Poincaré and others, have been shewn to be of great value in this respect.

90. It is necessary to notice very carefully the meaning attached to $\partial R/\partial a$ in equations (17). By means of the equations of Art. 50, R is expressed in terms of a, n, e, ϵ, ϖ, θ, i and

there exists between a, n the relation $n^2a^3=\mu$. It will be noticed that a only occurs as a coefficient and that n only occurs in the form $nt+\epsilon$. Hence we must not use the relation $n^2a^3=\mu$ *before* forming $\partial R/\partial a$ but differentiate with respect to a only as it occurs in R explicitly. In the canonical system of equations (19) this difficulty is not present.

The replacing of $nt+\epsilon$ by $\int ndt+\epsilon_1$ does not cause any trouble, since $\partial R/\partial\epsilon=\partial R/\partial\epsilon_1$.

91. Attention must be drawn to the meaning of the symbol δ as used in Arts. 73—75 and as used later. In the first case the variations for each element were *quite arbitrary* and it was therefore permissible to equate the coefficients of each of them to zero. Later they were the variations *actually taking place*, owing to the disturbing forces. Thus, when the variations were arbitrary, δR had a certain value depending on the arbitrary variations of the elements only; when the variations were the actual ones it was seen (Art. 86) that $\delta R=\frac{\delta R}{dt}dt=0$. This last equation is merely a direct consequence of the fact that R is a function of the coordinates only and not of the velocities and therefore that

$$\frac{\delta R}{dt}=\frac{\partial R}{\partial x}\frac{\delta x}{dt}+\frac{\partial R}{\partial y}\frac{\delta y}{dt}+\frac{\partial R}{\partial z}\frac{\delta z}{dt};$$

this expression is zero, since the velocities have the same form in disturbed and in undisturbed motion. This fact is used in Art. 86 to obtain the sixth equation when the other five have been found. It might equally have been used in Art. 80 to obtain $d\epsilon_1/dt$. The process would however have been somewhat longer.

92. The canonical system (19) is much more easily found by the method of Jacobi. In fact the natural way is to obtain these equations first and then to deduce the results of Arts. 83, 82. With the transformations given in Arts. 85, 86, it will be quite simple to reverse the process.

The equations (19) have been obtained by R. B. Hayward* in a direct manner.

The constants used may be defined geometrically and dynamically as follows:—

$-\alpha_1=$Time of passage through the nearer apse,

$\alpha_2=$Distance from node to perigee,

$\alpha_3=$Longitude of node;

$\beta_1=$Constant of Energy,

$\beta_2=$Twice area described in a unit of time in orbit,

$\beta_3=$ „ „ „ „ „ „ „ fixed plane.

(ii) The methods of Jacobi and Lagrange.

93. We shall now give a short account of the applications of the general dynamical methods of Hamilton, Jacobi and Lagrange to the problem of disturbed elliptic motion. In chronological order those of Lagrange should come first; their application to the discovery of equations (17) is however long and therefore his results will be stated only in so far as they are necessary

* "A direct demonstration of Jacobi's Canonical Formulæ," etc., *Quart. Jour. Math.* Vol. III. pp. 22–36.

for the explanation of Hansen's methods. The results of Jacobi's dynamical methods, which were based on those of Hamilton, will also be merely stated; references will be given to the more advanced treatises on Mechanics in which the proofs may be found.

94. *The Methods of Hamilton and Jacobi.*

Let T be the kinetic energy and F the force-function of a dynamical system. Suppose that there are n degrees of freedom and let $q_1, q_2 \dots q_n$ be the coordinates defining the position at time t. We suppose that there is no geometrical equation connecting the coordinates, that F is expressible in terms of the coordinates and of the time only, and that T does not contain the time explicitly.

Let the velocities be $\dot{q}_1, \dot{q}_2 \dots \dot{q}_n$; then T is a function of $q_i, \dot{q}_i \, (i = 1, 2 \dots n)$. Let

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \quad \dots\dots\dots\dots\dots\dots \text{(20)}:$$

the quantities p_i are called the generalised component momenta of the system or, more simply, the momenta.

Since T is a quadratic function of the velocities it can be also expressed as a quadratic function of the momenta in the form

$$T = A_{11}p_1^2 + 2A_{12}p_1p_2 + 2A_{13}p_1p_3 + \dots + A_{22}p_2^2 + \dots\dots\dots \text{(21)},$$

where A_{ij} is a function of the coordinates only.

Theorem I. The equations of motion may be put into the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where $H = T - F$*.

The *principal function* S is defined by the equation

$$S = \int^t (T + F)\, dt.$$

Suppose that the dynamical equations have been solved and that S has been expressed in terms of the coordinates, of the $2n$ necessary arbitrary constants (exclusive of the constant to be added to S by definition) and of the time. We then have

Theorem II.

$$\frac{\partial S}{\partial t} + T - F = 0, \qquad p_i = \frac{\partial S}{\partial q_i}.$$

* E. J. Routh, *Rigid Dynamics*, Vol. I. Art. 414.

Theorem III. S satisfies the partial differential equation.

$$\frac{\partial S}{\partial t} + A_{11}\left(\frac{\partial S}{\partial q_1}\right)^2 + 2A_{12}\frac{\partial S}{\partial q_1}\frac{\partial S}{\partial q_2} + \ldots - F = 0.$$

(This follows immediately from Theorem II. by the use of equation (21).)

Theorem IV. If, knowing only F and the coefficients A_{ij}, we can discover **any** *integral of this partial differential equation, involving n independent arbitrary constants $\beta_1, \beta_2 \ldots \beta_n$ (exclusive of that additive to S), of the form*

$$S = \phi(q_1, q_2, \ldots q_n, \beta_1, \beta_2, \ldots \beta_n, t);$$

the n complete integrals of the dynamical system will be given by the equations

$$\alpha_i = \frac{\partial \phi}{\partial \beta_i}, \qquad (i = 1, 2, \ldots, n)$$

the α_i being n new independent arbitrary constants.*

Solution of the Equations for Elliptic Motion by Jacobi's method.

95. We shall first apply these theorems to the problem of simple elliptic motion. There being three degrees of freedom, choose as coordinates the radius vector r, the longitude v reckoned on the fixed plane and the latitude U above this plane. We take the mass of the Moon for simplicity to be unity, so that $F = \mu/r$. The velocities $\dot{q}_i$ are $\dot{r}$, $\dot{v}$, $\dot{U}$ and

$$2T = \dot{r}^2 + (r^2 \cos^2 U)\,\dot{v}^2 + r^2 \dot{U}^2.$$

Hence from equation (20) the momenta will be

$$\dot{r} = \frac{\partial S}{\partial r}, \qquad (r^2 \cos^2 U)\,\dot{v} = \frac{\partial S}{\partial v}, \qquad r^2 \dot{U} = \frac{\partial S}{\partial U},$$

and therefore

$$2T = \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2 \cos^2 U}\left(\frac{\partial S}{\partial v}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial U}\right)^2.$$

The partial differential equation satisfied by S is then (Theorems II., III.)

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2 \cos^2 U}\left(\frac{\partial S}{\partial v}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial U}\right)^2\right] - \frac{\mu}{r} = 0.$$

We first require *some* integral of this involving three independent arbitrary constants (Theorem IV.).

* Id., Vol. II. Chap. X. Theorems II., III. are given fully. The result similar to Theorem IV. for the *characteristic* function only is proved, but the proof for the *principal* function is almost identical and may be easily reproduced.

96. To find one, assume

$$S = -\beta_1 t + \beta_3 v + S_1,$$

where S_1 contains neither v nor t and β_1, β_3 are arbitrary constants. Substituting, we obtain

$$\left(\frac{\partial S_1}{\partial r}\right)^2 + \frac{\beta_3^2}{r^2 \cos^2 U} + \frac{1}{r^2}\left(\frac{\partial S_1}{\partial U}\right)^2 - \frac{2\mu}{r} = 2\beta_1.$$

Let
$$\left(\frac{dS_2}{dr}\right)^2 = \frac{2\mu}{r} + 2\beta_1 - \frac{\beta_2^2}{r^2},$$

β_2 being a constant, so that

$$S_2 = \int^r dr \sqrt{\frac{2\mu}{r} + 2\beta_1 - \frac{\beta_2^2}{r^2}};$$

and assume
$$S_1 = S_2 + S_3,$$

where S_3 is independent of r. Then, from the equation for S_1, we have

$$\frac{\beta_3^2}{\cos^2 U} + \left(\frac{dS_3}{dU}\right)^2 = \beta_2^2;$$

whence
$$S_3 = \int^U dU \sqrt{\beta_2^2 - \beta_3^2/\cos^2 U}.$$

Substituting the values of S_2, S_3, S_1 in the assumed expression for S, an integral of the equation for S is therefore given by

$$S = -\beta_1 t + \beta_3 v + \int^r dr \sqrt{\frac{2\mu}{r} + 2\beta_1 - \frac{\beta_2^2}{r^2}} + \int^U dU \sqrt{\beta_2^2 - \frac{\beta_3^2}{\cos^2 U}}.$$

This contains three independent arbitraries β_1, β_2, β_3. The constant additive to S may be fixed by inserting any lower limits to the integrals. Let that of the second integral be 0 and that of the first r_a, where r_a is the smaller root of the equation

$$\frac{2\mu}{r} + 2\beta_1 - \frac{\beta_2^2}{r^2} = 0.$$

By Theorem IV. the integrals of the equations of motion are given by $\alpha_i = \partial S/\partial \beta_i$. Whence

$$\alpha_1 = -t + \int_{r_a}^r dr \left(\frac{2\mu}{r} + 2\beta_1 - \frac{\beta_2^2}{r^2}\right)^{-\frac{1}{2}},$$

$$\alpha_2 = -\int_{r_a}^r \frac{\beta_2}{r^2} dr \left(\frac{2\mu}{r} + 2\beta_1 - \frac{\beta_2^2}{r^2}\right)^{-\frac{1}{2}} + \int_0^U \beta_2 dU \left(\beta_2^2 - \frac{\beta_3^2}{\cos^2 U}\right)^{-\frac{1}{2}},$$

$$\alpha_3 = v - \int_0^U \frac{\beta_3 dU}{\cos^2 U}\left(\beta_2^2 - \frac{\beta_3^2}{\cos^2 U}\right)^{-\frac{1}{2}}.$$

The parts of α_1, α_2 due to the differentiation with respect to the limit r_a, are

$$-\frac{\partial r_a}{\partial \beta_i}\sqrt{\frac{2\mu}{r_a}+2\beta_1-\frac{\beta_2^2}{r_a^2}}, \qquad (i=1, 2)$$

and they therefore vanish by the definition of r_a.

97. It only remains now to connect the six constants α_1, α_2, α_3, β_1, β_2, β_3 with those ordinarily used in elliptic motion.

Let the two roots of the equation defining r_a be the greatest and least distances in the ellipse, that is $a(1+e)$, $a(1-e)$. We then have

$$a^2(1-e^2)=-\frac{\beta_2^2}{2\beta_1}, \qquad 2a=-\frac{\mu}{\beta_1},$$

whence $\beta_1=-\mu/2a, \qquad \beta_2=\sqrt{\mu a(1-e^2)}=h_0.$

Again, as $\beta_2^2-\beta_3^2/\cos^2 U$ must be always a positive quantity, we give to U its greatest value i and to β_3 a value such that the expression is then zero. Hence

$$\beta_3=\beta_2\cos i=h_0\cos i.$$

Further, α_1 is the value of $-t$ when $r=r_a$, that is, at perigee where the mean anomaly $nt+\epsilon-\varpi$ is zero. Hence $\alpha_1=(\epsilon-\varpi)/n$.

Also, α_3 is the value of v when $U=0$, that is, at the node. Hence $\alpha_3=\theta$.

Finally, we have

$$\int_0^U \beta_2 dU\left(\beta_2^2-\frac{\beta_3^2}{\cos^2 U}\right)^{-\frac{1}{2}}=\int_0^U\frac{\cos U dU}{(\sin^2 i-\sin^2 U)^{\frac{1}{2}}}.$$

Let $\sin U=\sin i\sin L$. Then (Fig. 5, Art. 73) since $U=M'M$, L is the angular distance from the node to the radius vector as in Art. 44; the value of the integral becomes L by the substitution. Hence α_2 is the value of L when $r=r_a$, that is, at perigee: therefore $\alpha_2=\varpi-\theta$.

The system of constants is then the same as that given by equations (18). It is now very easy to obtain the canonical system (19).

98. *Variation of Arbitrary Constants by Jacobi's method.*

We have by Theorems II., IV.

$$p_i=\frac{\partial S}{\partial q_i}, \qquad \alpha_i=\frac{\partial S}{\partial \beta_i}.$$

Since S is a function of the independent quantities α_i, q_i, these may be written in one equation,

$$\delta S=\Sigma(p\delta q+\alpha\delta\beta) \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

where $\Sigma p\delta q = p_1\delta q_1 + \ldots + p_n\delta q_n$, etc.; the operator δ may denote any variation whatever.

The Hamiltonian equations are

$$\dot{q}_i = \frac{\partial}{\partial p_i}(T - F), \qquad -\dot{p}_i = \frac{\partial}{\partial q_i}(T - F).$$

Suppose we put $F = \mu/r + R$. Let the values of q_i, p_i, already obtained for the case $R = 0$, be made to satisfy the equations when R is not zero, by considering α_i, β_i variable. (This is merely another example of coordinates and velocities having the same form in two problems when they shall have been expressed in terms of the arbitraries and of the time.)

Let Δq_i, Δp_i be the small increments to be added to q_i, p_i in time dt, due to the presence of R. Then from the Hamiltonian equations we have, since T is unaltered,

$$\Delta q_i = -\frac{\partial R}{\partial p_i}dt, \qquad \Delta p_i = \frac{\partial R}{\partial q_i}dt.$$

These, expressed in one equation, give*

$$dt\delta R = \Sigma(\Delta p\delta q - \Delta q\delta p) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23).$$

Again, as δ denotes *any* increment, it may have the value Δ so that, from equation (22),

$$\Delta S = \Sigma(p\Delta q + \alpha\Delta\beta).$$

Whence $\qquad \Sigma\delta(p\Delta q + \alpha\Delta\beta) = \delta\Delta S = \Delta\delta S = \Sigma\Delta(p\delta q + \alpha\delta\beta).$

Therefore, as $\delta(p\Delta q) = \delta p\Delta q + p\Delta\delta q$, etc., this equation gives

$$\Sigma(\delta\alpha\Delta\beta - \delta\beta\Delta\alpha) = \Sigma(\delta q\Delta p - \delta p\Delta q) = dt\delta R,$$

by (23).

Finally, as Δ denotes the actual increment due to the presence of R, we have

$$\Delta\alpha = \frac{d\alpha}{dt}dt, \qquad \Delta\beta = \frac{d\beta}{dt}dt.$$

Therefore, substituting in the equation just obtained,

$$\delta R = \Sigma\left(\delta\alpha\frac{d\beta}{dt} - \delta\beta\frac{d\alpha}{dt}\right),$$

or,
$$\frac{\partial R}{\partial \alpha_i} = \frac{d\beta_i}{dt}, \qquad \frac{\partial R}{\partial \beta_i} = -\frac{d\alpha_i}{dt},$$

where R is now supposed expressed in terms of t, α_1, α_2, ... β_1, β_2 ....

* Here $\Delta p\delta q$ denotes the product of Δp and δq, and so elsewhere.

From this result it is evident that Jacobi's method of solution produces a system of canonical constants. In the case of disturbed elliptic motion, we shall therefore have as one system the values of α_i, β_i given in Art. 97. From the equations just found we can deduce (17) by reversing the processes of Arts. 84—86.

Lagrange's Method.

99. Suppose for the sake of simplicity that in a dynamical problem there are three degrees of freedom and that the complete integrals are

$$q_i = \phi_i(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, t), \quad p_i = \phi_i'(\gamma_1, \gamma_2, \ldots \gamma_6, t), \quad (i = 1, 2, 3)$$

where q, p are defined as before and the system of constants of solution $\gamma_1 \ldots \gamma_6$ is quite arbitrary. Since the constants are independent, we may suppose them determined in terms of the coordinates and momenta by equations of the form

$$\gamma_i = \psi_i(q_1, q_2, q_3, p_1, p_2, p_3, t), \qquad (i = 1, 2 \ldots 6).$$

Let now R be added to the force function and let the solution be made to retain the same form by considering the arbitraries γ_i as variable. Lagrange has shown that *the six equations which determine the* γ_i *are*

$$\left.\begin{aligned}\frac{d\gamma_1}{dt} &= \qquad\qquad\qquad (\gamma_1, \gamma_2)\frac{\partial R}{\partial \gamma_2} + (\gamma_1, \gamma_3)\frac{\partial R}{\partial \gamma_3} + \ldots + (\gamma_1, \gamma_6)\frac{\partial R}{\partial \gamma_6}\\ \frac{d\gamma_2}{dt} &= (\gamma_2, \gamma_1)\frac{\partial R}{\partial \gamma_1} \qquad\qquad + (\gamma_2, \gamma_3)\frac{\partial R}{\partial \gamma_3} + \ldots + (\gamma_2, \gamma_6)\frac{\partial R}{\partial \gamma_6}\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \ldots (24),$$

where
$$(\gamma_i, \gamma_j) = \left(\frac{\partial \gamma_i}{\partial q_1}\frac{\partial \gamma_j}{\partial p_1} - \frac{\partial \gamma_i}{\partial p_1}\frac{\partial \gamma_j}{\partial q_1}\right) + \ldots + \left(\frac{\partial \gamma_i}{\partial q_3}\frac{\partial \gamma_j}{\partial p_3} - \frac{\partial \gamma_i}{\partial p_3}\frac{\partial \gamma_j}{\partial q_3}\right).$$

It is evident that after the differentiations have been carried out, the coefficients (γ_i, γ_j) can be expressed in terms of the arbitraries and of the time. Lagrange has shown however that, when so expressed, *t is not present explicitly in any of these coefficients*, so that the equations (24) only contain the time explicitly through its presence in $\partial R/\partial \gamma_i$*. The equations (17) are a particular case of these results and were so obtained by Lagrange†; the original problem is that of undisturbed elliptic motion, R is the disturbing function and to the arbitraries γ_i, are given the values $a, e \ldots \theta$.

It is easy to see that any function λ of the γ_i which does not contain t explicitly, may replace one of the γ_i, say γ_1, and that the equations

* Proofs of these results are given by Routh, *Rigid Dynamics*, Vol. II. Arts. 477, 478 and by Cheyne, *Planetary Theory*, Appendix.

† *Méc. Anal.*, Pt. II. Section VII. Chap. ii. See also Tisserand, *Méc. Cél.* Vol. I. Chap. X. and O. Dziobek, *Math. The. der Planeten-Bewegungen*, §§ 10, 11.

(24) will be still true if in them we replace γ_1 by λ: for the system of constants γ_i was arbitrary. The disturbing function is then supposed to be expressed in terms of $\lambda, \gamma_2, \gamma_3 \dots \gamma_6, t$.

100. The transformation of the equations for $\gamma_1, \gamma_2 \dots$ to those for $\lambda, \gamma_2 \dots$ might also have been made directly by means of the assumed relation between λ and the γ_i. This way of looking at the problem enables us to give an extension to the meaning of λ. If we define λ by equations of the form

$$\lambda = \Sigma \int A_i d\gamma_i \quad \text{or} \quad d\lambda = \Sigma A_i d\gamma_i,$$

where the A_i are functions of the γ_i only—although the expression $\Sigma A_i d\gamma_i$ may not be a perfect differential, the equations corresponding to (24) for $\lambda, \gamma_2, \dots$ will still hold because the direct transformation only involves the *differentials* of the arbitraries.

When R is not expressible explicitly in terms of $\lambda, \gamma_2, \dots$, the expression $\partial R/\partial \lambda$ may be defined by the equation

$$\frac{\partial R}{\partial \lambda} = \Sigma A_i \frac{\partial R}{\partial \gamma_i},$$

that is, $\partial R/\partial \lambda$ has the same meaning as if R had been expressible in terms of the new arbitraries. With this convention it will be unnecessary to make a direct transformation.

101. Elements defined in this latter way have been called by Jacobi *pseudo-elements**.

Hansen defines *ideal coordinates* to be such that they and their first differentials with respect to the time have the same form whether the motion be disturbed or undisturbed*. Such are r, r_1, v, x, y, z, etc., these being all referred to fixed axes. We can however have ideal coordinates referred to moving axes.

Consider a set of rectangular axes of which those of X, Y are in the plane of the orbit and that of Z is perpendicular to it. Let the axis of X be placed at a departure point. Let

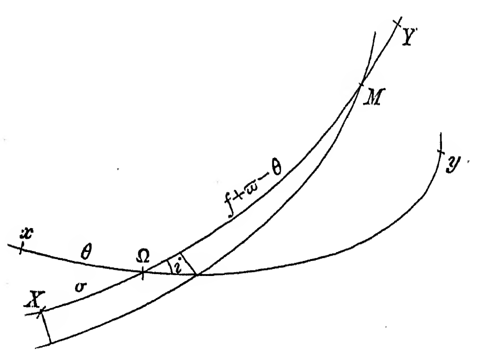

Fig. 6.

* See the letters of Hansen and Jacobi referred to at the end of Art. 102.

$v_1 = XM =$ the longitude of the Moon reckoned from the departure point. Then (Art. 79) v_1 and dv_1 have the same form whether the motion be disturbed or undisturbed; v_1 is therefore an ideal coordinate. And yet v_1 is not expressible in terms of the elements and of the time unless one of these be a pseudo-element. For if $X\Omega = \sigma$, we have from the figure, since the line joining two consecutive positions of X is perpendicular to ΩX,

$$d\sigma = \cos i \, d\theta, \quad v_1 = f + \varpi - \theta + \sigma.$$

If then v_1 be one of the coordinates in terms of which R is expressed, there will be present in R the pseudo-element σ.

102. Let us see how R will be expressed when these axes are used. Euler's formulæ of transformation of the coordinates of a point (xyz) to (XYZ) are*

$$x = a_1 X + b_1 Y + c_1 Z, \qquad X = a_1 x + a_2 y + a_3 z,$$
$$y = a_2 X + b_2 Y + c_2 Z, \qquad Y = b_1 x + b_2 y + b_3 z,$$
$$z = a_3 X + b_3 Y + c_3 Z, \qquad Z = c_1 x + c_2 y + c_3 z,$$

where $(a_1 b_1 c_1)$, $(a_2 b_2 c_2)$, $(a_3 b_3 c_3)$ are the direction cosines of the axes of X, Y, Z, referred to those of x, y, z: they are trigonometrical functions of σ, θ, i. If (XYZ) be the coordinates of the Moon, we have $Z=0$. Hence R is expressible in terms of X, Y, σ, θ, i, or in terms of r, v_1, σ, θ, i.

Here the differentials of R with respect to v_1, σ have a meaning without further definition. For dv_1 is the angle between two consecutive positions of the radius vector reckoned in the plane of the orbit and therefore $\partial R / r\partial v_1 = \mathfrak{T}$, $\partial R/\partial r = \mathfrak{P}$; the force perpendicular to the plane of the orbit will now depend on the differentials of R with respect to σ, θ, i (see Chap. x.).

The use of the pseudo-element σ introduces another arbitrary constant, namely, the value of σ at the origin of time.

The general condition that X, Y, Z may be ideal coordinates when the new rectangular axes are any whatever, is

$$x\,da_1 + y\,da_2 + z\,da_3 = 0,$$
$$x\,db_1 + y\,db_2 + z\,db_3 = 0,$$
$$x\,dc_1 + y\,dc_2 + z\,dc_3 = 0;$$

for then dX, dY, dZ will have the same form whether a_1, b_1 ... be constant or variable. These three conditions, involving only the differentials of a_1, a_2,... are available whether the elements be true elements or pseudo-elements.

On the subjects of Arts. 100–102, see

Lagrange, *Méc. Anal.* Pt. II. Sec. VII. No. 70.

Binet, "Sur la Variation des Constantes Arbitraires," *Journal de l'École Polytechnique*, Vol. XVII. p. 76.

Hansen, "Auszug eines Schreibens," etc.; Jacobi, "Auszug zweier Schreiben," etc. *Crelle*, Vol. XLII. pp. 1–31.

Hansen, "Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten," *Abh. d. K. Sächs. Ges. d. Wissensch.* Vol. V. pp. 41—218.

* P. Frost, *Solid Geometry* (1875), Art. 146.

Cayley, "A Memoir on Disturbed Elliptic Motion," *Mem. R. A. S.* Vol. XXVII. pp. 1–29; *Coll. Works*, Vol. III. pp. 270–292.

Donkin, "On the Differential Equations of Dynamics," *Phil. Trans. R. S.* 1855, Pt. II. pp. 352–354.

103. We can very quickly deduce a set of canonical equations from the formulæ (24).

Let $\gamma_1, \gamma_2, \gamma_3$ be defined as the values of q_1, q_2, q_3 at time $t=\tau$ and $\gamma_4, \gamma_5, \gamma_6$ those of p_1, p_2, p_3 at the same time. Since, in the process of forming the partials $\frac{\partial \gamma_i}{\partial p_1}, \frac{\partial \gamma_i}{\partial q_1}$ etc., it makes no difference whether t be constant or variable, and since t ultimately disappears from the coefficients (γ_i, γ_j), we can give to t the value τ, that is, we can put $q_1=\gamma_1$ etc., *before* forming these coefficients. We shall then have

$$(\gamma_i, \gamma_j)=\left(\frac{\partial \gamma_i}{\partial \gamma_1}\frac{\partial \gamma_j}{\partial \gamma_4}-\frac{\partial \gamma_i}{\partial \gamma_4}\frac{\partial \gamma_j}{\partial \gamma_1}\right)+\dots.$$

Whence, since all the arbitraries are independent, we obtain

$$(\gamma_1, \gamma_4)=1=-(\gamma_4, \gamma_1), \quad (\gamma_2, \gamma_5)=1=-(\gamma_5, \gamma_2), \quad (\gamma_3, \gamma_6)=1=-(\gamma_6, \gamma_3);$$

and all the other coefficients (γ_i, γ_j) will be zero.

Denote the values of the coordinates and momenta at time τ by Q_i, P_i. We have in the present case $\gamma_1=Q_1 \dots$, $\gamma_4=P_1 \dots$ and therefore equations (24) become

$$\frac{dQ_i}{dt}=\frac{\partial R}{\partial P_i}, \qquad \frac{dP_i}{dt}=-\frac{\partial R}{\partial Q_i}.$$

Hence the values of the coordinates and momenta at a given time form a system of canonical constants; P_i, Q_i are considered here as the arbitraries of the original solution and R is supposed to be expressed in terms of them and of the time. This system of canonical constants was first given by Lagrange.

104. The disturbed values of Q_i, P_i are given by

$$Q_i=\int \frac{\partial R}{\partial P_i}dt, \qquad P_i=-\int \frac{\partial R}{\partial Q_i}dt,$$

the lower limits of the integrals giving new arbitraries which are absolute constants. Suppose the integrations have been performed by any process so as to give the disturbed values of P_i, Q_i; these latter will then be functions of t, τ and of absolute constants. But since the results must hold for every value of τ, that is, at any point in the orbit, we shall get the disturbed values of the coordinates and of the momenta at time t by putting $\tau=t$ in these equations. Whence

$$q_i=\overline{\int \frac{\partial R}{\partial P_i}dt}, \qquad p_i=-\overline{\int \frac{\partial R}{\partial Q_i}dt};$$

where the bar denotes that τ has been changed into t after the integrations have been performed.

This extension is due to Hansen*. It may also be written

$$q_i=\overline{\int \frac{dQ_i}{dt}dt}, \qquad p_i=\overline{\int \frac{dP_i}{dt}dt},$$

* "Commentatio de corporum coelestium perturbationibus," *Astr. Nach.* Vol. XI. Col. 322.

where, under the integral sign, we suppose disturbed values substituted. From the former of these results the general theorem, which lies at the basis of all Hansen's researches into the lunar and the planetary theories, can be deduced. As however the form in which he uses it can be exhibited as an elementary result of the integral calculus, it will not be proved here.

The theorem in question is constructed to prove that any function of the elements and of the time may be differentiated, the disturbed values of the elements substituted and the result integrated, with the time—as far as it occurred explicitly in the function—constant during the whole process*.

105. The earlier literature on the general dynamical principles of Lagrange, Hamilton and Jacobi and on their applications to the subject of this chapter, is very large. It has been collected and a summary of the results is given by

Cayley, "Report on the recent Progress of Theoretical Dynamics," *B. Ass. Rep.* 1857; *Coll. Works*, Vol. III. pp. 156–204. See also "Report on the Progress of the solution of certain problems in Dynamics," *B. Ass. Rep.* 1862; *Coll. Works*, Vol. IV. pp. 514, 515.

The chief original memoirs are to be found as follows:—

Lagrange, *Méc. Anal.*

Poisson, "Mém. sur la variation," etc., *Jour. de l'Éc. Poly.* Vol. VIII. pp. 266–344.

Hamilton, "On a general method in Dynamics," etc., *Phil. Trans. R. S.* 1834, pp. 247–308; 1835, pp. 95–144.

Jacobi, *Vorlesungen über Dynamik.*

The following treatises may also be consulted with great advantage:

Thomson and Tait, *Natural Philosophy*, Vol. I. Chap. II.

Tisserand, *Méc. Cél.* Vol. I. Intro. and Chap. IX.

Dziobek, *Math. The.*, etc., Abschnitt II.

Poincaré, *Méc. Cél.* Vol. I. Chap. I.

* *Fundamenta*, pp. 22–25. A proof by Taylor's Theorem is given in the Commentatio, etc. Cols. 323–326.

CHAPTER VI.

THE DISTURBING FUNCTION.

106. In the equations of motion obtained in Chapter II. we have expressed the forces in terms of the partial differential coefficients of R or F. In order to obtain the forces in terms of the variables, R must be suitably expressed. The object of this Chapter is to find expressions of such a form that the labour of making the developments may be as small as possible.

We have seen that with the methods of procedure usually adopted in the lunar theory, the second approximation to the values of the coordinates is obtained by substituting the results of the first approximation in the terms previously neglected. In general, the first approximation being an ellipse, this amounts to expressing the disturbing forces in terms of the elliptic elements and of the time.

Now the determination of motion in space requires a knowledge of three component forces. If we form these forces directly from the general expression of R in terms of the coordinates—a process easily performed—and then develope the results in terms of the elliptic elements and of the time, there will be *three* developments to be made. To save this labour we develope R in terms of the elements and of the time; the forces can then be deduced by transforming their differentials with respect to the coordinates into differentials with respect to certain functions of the elements and of the time which occur explicitly in the development of R.

The principal object is then to develope R in terms of the time and of the elliptic elements of the orbits of the Moon and the Sun. According to the different functions of the elements used, there will be slightly different forms of expression. They can, however, be all deduced from those given in Section (iii) which contains Hansen's method. In connection with de Pontécoulant's method, some general properties of the disturbing function will be given. The variety of forms by which R can be expressed arises from the fact that R depends only on r, r' and on the cosine of the angle between r, r'.

107. In the lunar theory, as already stated in Art. 9, we always begin by expanding the disturbing function in powers of r/r'. We have (Art. 8),

$$R = \frac{m'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}} - m'\frac{xx'+yy'+zz'}{r'^3}.$$

Let S be the cosine of the angle between the radii vectores of the Sun and the Moon. Then $$xx'+yy'+zz' = rr'S,$$

and $$R = \frac{m'}{r'}\left\{\left(1 - 2\frac{r}{r'}S + \frac{r^2}{r'^2}\right)^{-\frac{1}{2}} - \frac{r}{r'}S\right\}.$$

Expand the first term of this expression in powers of r/r' by means of the Binomial Theorem or by the use of Legendre's coefficients *. The first term, which is m'/r', may be omitted since it does not contain the coordinates of the Moon; the second term $m'rS/r'^2$, will be cancelled by the term $-m'rS/r'^2$.

We therefore obtain

$$R = \frac{m'r^2}{r'^3}\left\{(\tfrac{3}{2}S^2 - \tfrac{1}{2}) + \frac{r}{r'}(\tfrac{5}{2}S^3 - \tfrac{3}{2}S) + \frac{r^2}{r'^2}(\tfrac{35}{8}S^4 - \tfrac{15}{4}S^2 + \tfrac{3}{8}) \right.$$
$$\left. + \frac{r^3}{r'^3}(\tfrac{63}{8}S^5 - \tfrac{35}{4}S^3 + \tfrac{15}{8}S) + \dots\right\} \dots\dots(1).$$

(i) *Development of R necessary for the solution of Equation (A), Chapter II. The Properties of R.*

108. The first process is to develope R in terms of r, v, s, r', v'. Let m' be the place on xy (fig. 4, Art. 44) where the radius vector of the Sun cuts the unit sphere. According to the notation previously used, we denote by v, v' the true longitudes of the Moon and the Sun reckoned from x, and by s the latitude of the Moon above the plane of xy.

From the right-angled triangle $MM'm'$ we have,

$$\cos(v - v') = \cos M'm' = \cos Mm'/\cos M'M = S\sqrt{(1+s^2)}.$$

Hence $$S = \frac{\cos(v-v')}{\sqrt{(1+s^2)}} = (1 - \tfrac{1}{2}s^2 + \tfrac{3}{8}s^4 - \dots)\cos(v - v') \dots\dots\dots\dots(2).$$

Substituting this value of S in (1) we have, neglecting s^4, $m'r^4/r'^5$ and

* Todhunter, *Functions of Laplace, Lamé and Bessel*, Chap. I.

higher powers of s^2, r/r', and replacing powers of cosines by cosines of multiples of $(v-v')$,

$$R = \frac{m'r^2}{r'^3}\left\{\tfrac{1}{4} + \tfrac{3}{4}(1-s^2)\cos 2(v-v') - \tfrac{3}{4}s^2\right\}$$
$$+ \frac{m'r^3}{r'^4}\left\{\tfrac{3}{8}(1-\tfrac{11}{2}s^2)\cos(v'-v) + \tfrac{5}{8}(1-\tfrac{3}{2}s^2)\cos 3(v-v')\right\}$$
$$+ \dots\dots \quad (3).$$

109. We have now to express R in terms of the time and of the elements of the elliptic orbits of the Moon and the Sun, according to the principles laid down in Chapter IV. Before doing so it is necessary to know something further about the numerical values of e, e', γ, a/a' (in powers of which the expansions will be made), in order that we may have some idea of the number of terms necessary to secure a given degree of accuracy in the results. We ought strictly to know the meanings to be attached to these constants when the motion is disturbed; but since in any of the systems used to fix their meaning, the numerical values only vary to a slight extent, for the purposes in view here it is sufficient to give a general idea of their magnitude in the case of the Moon.

The most important ratio is that of the mean motions n', n. It does not occur directly in the expansion of R; it will be seen, however, in Art. 114, that n'^2/n^2 is a factor of R. The numerical values are approximately,

$$\frac{n'}{n} = \tfrac{1}{13}, \quad e = \tfrac{1}{20}, \quad e' = \tfrac{1}{60}, \quad \gamma = \tfrac{1}{11}, \quad \frac{a}{a'} = \tfrac{1}{400}.$$

We consider n'/n to be a small quantity of the first order. Consequently n'/n, e, e', γ are small quantities of the first order and a/a' is one of the second order.

On the basis that 1/13 is of the first order, $e'^2 = \frac{1}{3600}$ would be of the third order, $e'a/a' = \frac{1}{24000}$ of the fourth order, and so on. But for simplicity we shall consider them of the order denoted by the index. Hence $(n'/n)^{p_1}e^{p_2}e'^{p_3}\gamma^{p_4}(a/a')^{p_5}$ will be said to be of the order $p_1+p_2+p_3+p_4+2p_5$.

110. The equations of Art. 50 in which w, η_0 are, by Art. 68, replaced by $cnt+\epsilon-\varpi=\phi$, $gnt+\epsilon-\theta=\eta$, give the values of r, v, s in terms of the elements a, n, e, ϵ, ϖ, θ, γ and of the time. If, in the same equations, we accent the letters and put $c'=1$, $g'=1$, they will give the values of r', v', s' in terms of the time and of the elliptic elements a', n', e', ϵ', ϖ', θ', γ'. But since the Sun's orbit is in the plane of reference, $\gamma'=0$, $s'=0$: with these values θ' disappears. Substituting in R, the disturbing function will be found expressed in terms of t and of the elements a, n, e, ϵ, ϖ, θ, γ, a', n', e', ϵ', ϖ'.

The form of the development of R.

111. Let $$\xi = (n - n')t + \epsilon - \epsilon'.$$

Since v, v' only occur in R in the form $\cos p(v - v')$, (p any integer) and as (Art. 50),

$$v = nt + \epsilon + A, \quad v' = n't + \epsilon' + A',$$

where A and A' consist only of periodic terms depending on the arguments ϕ, η and $\phi'(= n't + \epsilon' - \varpi')$ respectively, we have

$$\cos p(v - v') = \cos p\xi \cos p(A - A') - \sin p\xi \sin p(A - A').$$

Also, A, A' being small quantities of the first order at least, we suppose that expansions in powers of A, A' are possible. Hence

$$\cos p(v-v') = \left\{1 - \frac{p^2(A-A')^2}{2\,!} + \ldots\right\}\cos p\xi - \left\{\frac{p(A-A')}{1\,!} - \frac{p^3(A-A')^3}{3\,!} + \ldots\right\}\sin p\xi.$$

Therefore, all the terms arising from v, v' can be expressed by means of cosines of sums of multiples of the angles ξ, ϕ, 2η, ϕ'.

Finally, r, r' and s^2 being expressible in terms of ϕ, ϕ' and of ϕ, 2η, respectively, *R can be expressed by a series of cosines of sums of multiples of the* **four** *angles ξ, ϕ, ϕ', 2η, with coefficients depending on m', a, e, γ^2, a', e'.*

112. Owing to the introduction of c and g, the coefficients of t in these arguments and in all the arguments which are present in R, will never vanish unless the argument itself vanishes. For these coefficients of t will all be linear functions with integral coefficients, of $n - n'$, cn, n', gn, that is, of n, n', cn, gn; it will be seen later that c, g are not in general commensurable with an integer or with one another, and n'/n was assumed to be an incommensurable ratio. Hence no linear relation with integral coefficients will exist.

113. *The connection between the arguments and the coefficients.*

The constant γ enters into R only through its presence in v, s. Since only even powers of s are present in R, a glance at the equations of Art. 50 will show that *only even powers of γ are present in R.*

Also, if we leave aside the factor $m'a^2/a'^3$ which arises from $m'r^2/r'^3$, equation (1) shows that even powers of a/a' in the coefficient of any term will accompany even multiples of $v - v'$ and therefore of ξ in the argument of that term; similarly, odd powers of a/a' accompany odd multiples of ξ.

Combining these results with those of Arts. 40, 47, 111, we see
(a) that the arguments of all terms in R are of the form

$$j\xi \pm p\phi \pm p'\phi' \pm 2q\eta, \quad (j, p, p', q = 0, 1, 2\ldots);$$

(*b*) that the coefficient of the term having this argument is at least of the order

$$e^p e'^{p'} \gamma^{2q} \text{ or } e^p e'^{p'} \gamma^{2q} \frac{a}{a'},$$

according as j is even or odd;

(*c*) that any term in the coefficient is of the order

$$e^{p_1} e'^{p_1'} \gamma^{2q_1} (a/a')^{j_1},$$

where p_1, p_1', q_1 are respectively equal to p, p', q or are greater than them by even integers, and j, j_1 are odd or even together.

The factor $e^p e'^{p'} \gamma^{2q}$ (or $e^p e'^{p'} \gamma^{2q} a/a'$) which occurs in the coefficient of the term with the above argument may be called the *characteristic part of the coefficient* or, more simply, the *characteristic*.

De Pontécoulant's expansion for R.

114. Since $m' = n'^2 a'^3$, $\mu = n^2 a^3$, we have

$$\frac{m'a^2}{a'^3} = \frac{\mu}{a}\frac{n'^2}{n^2} = \frac{\mu}{a} m^2, \text{ where } m = \frac{n'}{n}.$$

It will be found convenient in de Pontécoulant's theory to choose the units of mass, length and time so that $\mu = 1$. With these units, m' is the ratio of the mass of the Sun to the sum of the masses of the Earth and the Moon. We can now put $n^2 a^3 = 1$, and

$$\frac{m'a^2}{a'^3} = \frac{m^2}{a} \quad \text{.................................(4).}$$

The development of R, complete as far as the first order in e, e', γ^2, a/a', is given below; for the sake of illustration, some terms of higher orders are included. The shortest method of actually performing the expansions will be explained in Arts. 124—126.

$$\begin{aligned} R = \frac{m^2}{a} \Big[\; & \tfrac{1}{4} + \tfrac{3}{4}\cos 2\xi \\ & - \tfrac{1}{2}e\cos\phi - \tfrac{9}{4}e\cos(2\xi - \phi) + \tfrac{3}{4}e\cos(2\xi + \phi) \\ & + \tfrac{3}{4}e'\cos\phi' + \tfrac{21}{8}e'\cos(2\xi - \phi') - \tfrac{3}{8}e'\cos(2\xi + \phi') \\ & - \tfrac{3}{8}\gamma^2 - \tfrac{3}{8}\gamma^2\cos 2\xi + \tfrac{3}{8}\gamma^2\cos 2\eta + \tfrac{3}{8}\gamma^2\cos(2\xi - 2\eta) \\ & + \tfrac{3}{8}\frac{a}{a'}\cos\xi + \tfrac{5}{8}\frac{a}{a'}\cos 3\xi \\ & + \tfrac{3}{8}e^2 - \tfrac{1}{8}e^2\cos 2\phi + \ldots + \tfrac{3}{8}e'^2 + \tfrac{9}{8}e'^2\cos 2\phi' + \tfrac{51}{8}e'^2\cos(2\xi - 2\phi') \\ & + \tfrac{3}{8}\frac{a}{a'}e'\cos(\xi + \phi') - \tfrac{15}{16}\frac{a}{a'}ee'\cos(\xi - \phi + \phi') + \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big] \ldots\ldots\ldots\ldots (5). \end{aligned}$$

To deduce the disturbing forces.

115. We have now to form $\dfrac{\partial R}{\partial r}, \dfrac{\partial R}{\partial v}, \dfrac{\partial R}{\partial s}$. Since the disturbing function has been expressed in terms of the elliptic elements, these partial differentials must be transformed so that we can deduce the functions which they represent directly from (5). For the purposes of this and of the next article, the factor m^2/a must be supposed to be replaced in (5) by its value $m'a^2/a'^3$.

In the first place, since a only enters into R explicitly through r, and since r is of the form $a(1+\rho)$, where ρ is independent of a, we have

$$r\frac{\partial R}{\partial r} = a\frac{\partial R}{\partial a}.$$

Here $\partial R/\partial a$ has the meaning assigned in Art. 90. Whence, if we consider only the terms which have a^2 as a factor,

$$r\frac{\partial R}{\partial r} = a\frac{\partial R}{\partial a} = 2R \text{ ....(6).}$$

Similarly for those which have a^3 as a factor,

$$r\frac{\partial R}{\partial r} = 3R \text{ ....(7),}$$

and so on.

Secondly, since v occurs only in the form $v-v'$, and since ξ only arises in R through the first term of the substitution of $\xi + A - A'$ for $v-v'$, we have

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial \xi} \text{ ....(8).}$$

Thirdly, it is found to be simpler to deduce $\partial R/\partial s$ directly from the equation (3) and then to substitute elliptic values for the coordinates. No transformation will therefore be necessary.

116. Some further properties may be noted. We have from the definition of $d'R$ (Art. 12), since R is a function only of the coordinates of the Sun and of the Moon,

$$\frac{d'R}{dt} = \frac{dR}{dt} - \frac{\partial R}{\partial r'}\frac{dr'}{dt} - \frac{\partial R}{\partial v'}\frac{dv'}{dt}.$$

If we regard the first term of R only,

$$r'\frac{\partial R}{\partial r'} = -3R,$$

and generally,

$$\frac{\partial R}{\partial v'} = -\frac{\partial R}{\partial v} = -\frac{\partial R}{\partial \xi} \quad\text{.............................(8').}$$

Hence
$$\int d'R = R + 3\int R\frac{dr'}{r'} + \int \frac{\partial R}{\partial \xi} dv' \quad\text{....................(9),}$$

a form which is frequently of value. If we are considering the term of R which contains the factor $m'r^{p+2}/r'^{p+3}$, instead of 3 we must put $p+3$.

By means of this result we only need to form the single differential $\partial R/\partial \xi$ in the radius and longitude equations (A), Chap. II., when R has been found.

It is easy to see the truth of the statement made in Art. 65, that $\partial R/\partial v$, $d'R/dt$ will contain no constant terms. For R contains only constant terms and cosines and therefore $\partial R/\partial v = \partial R/\partial \xi$ only sines of angles without any constant term. Also in (9), R, r', dv' are expressible by means of cosines and constant terms while dr', $\partial R/\partial \xi$ consist of sines only, whence $d'R/dt$ contains no constant term. All the functions we have to deal with are expressible either by means of cosines and constant terms or by means of sines with or without a term of the form $t \times$ const. + const.

The effect produced on the orders of the coefficients by the integration of the equations (A).

117. The substitution of n^2/a for $m'a^2/a'^3$ shows that the coefficient of every term in R is at least of the second order of small quantities. It does not however follow that the corresponding terms in the expressions for the coordinates are of the same orders as the terms in R from which they arise. The integrations will, in certain cases, cause small divisors to appear which will lower the orders of the coefficients to which they are attached.

We have seen in Art. 66, that a term of the form

$$A\cos(kt+\alpha)$$

present in the right-hand members of any of the three equations (A) will produce terms in r, s of the form

$$\frac{A}{-k^2+n^2}\cos(kt+\alpha),$$

and it is evident that it will produce in v a term of the form

$$\frac{A}{k}\sin(kt+\alpha).$$

There are three cases to be considered, depending on the magnitude of k.

(*a*) If k be a small quantity of the first order, the terms in r, s will be of the same order as A and the term in v will have its coefficient lowered one order.

(*b*) If $k^2 - n^2$ be a small quantity of the first order, the terms in r, s will have the corresponding coefficients lowered one order, while the order of the coefficient of the term in v remains unaltered.

Further, in the longitude equation there occurs the integral $\int \frac{\partial R}{\partial v} dt$, and in the radius-vector equation the integral $\int d'R$.

If a term of the form $A \sin(kt + \alpha)$ is present in $\partial R/\partial v$ and if k be of the first order, the coefficient of the term will be lowered two orders by the integration of the longitude equation. If the term occur in $d'R/dt$, its coefficient will be lowered one order by the integration of the radius vector equation.

(*c*) There is one term in R for which $k = n$, namely, the term with argument $\xi + \phi'$; its coefficient is of the order $m^2 e' a/a'$. Contrary to what might have been expected from the remarks of Art. 66, this term does not cause t to appear as a factor of the coefficient. The argument, expressed in terms of the elements, is

$$nt + \epsilon - n't - \epsilon' + n't + \epsilon' - \varpi' = nt + \epsilon - \varpi'.$$

To understand this, it is necessary to refer to Art. 67 where it was seen that the first approximation could only be obtained in a suitable form by supposing certain terms of the disturbing function (which should, by the method of continued approximation, have been neglected) to be included. It was seen that instead of the equation $\ddot{x} + n^2 x = Q$, the more correct equation to deal with is $\ddot{x} + (n^2 + b_1)\, x = Q$, where b_1 and Q are small quantities arising from the disturbing function. The first approximation (that is, the Complementary Function) then consisted of terms of period $2\pi/cn$. If, with this first approximation, Q be expressed in terms of the time and if a term $A \cos(nt + \alpha)$ arises from Q, we see that no modification is necessary, since its period is $2\pi/n$ and not $2\pi/cn$: further, no terms proportional to the time will arise. Finally a term $A' \cos(cn + \alpha')$ in Q will cause no difficulty owing to the definition of c.

The terms for which k is small are known as *long-period inequalities.* Their effect is in general most marked on the longitude. The terms for which k is numerically nearly equal to n, are those whose periods nearly coincide with the mean period. They produce marked effects on the radius vector and latitude and thence on the longitude.

118. Let us examine the case (*c*) of the last article a little more closely and see in what way the ordinary method of approximation may be applied to a term of the form considered.

The equation for the second approximation to r can be put into the form (see Art. 130),

$$\ddot{x}+n^2x+b_1x+b_2x^2+b_3x^3+\ldots=Q,$$

where b_1, b_2, ..., Q are the portions arising from the action of the Sun which, when the results of the first approximation are substituted, consist entirely of known terms.

In dealing with the second approximation we neglect x^2, x^3,... and substitute the results of the first approximation in b_1x, Q, so that a term of the form $A\cos(nt+\alpha)$ in $Q-b_1x$ appears to give an infinite value to the coefficient of the corresponding term in x. But we have seen that this is not really so and that the coefficient can only be found by including in the second approximation, terms of higher orders. It is the simplest plan, in actual calculation, to leave this coefficient indeterminate until the third approximation is reached: it can then be found because, in the third approximation, the results of the second approximation, substituted in x^2, x^3,... will produce terms of this form and these can be equated to the corresponding terms in x, Q.

It is not difficult to see how a term with argument $\xi+\phi'$ and with a known coefficient may arise in x^2 in the third approximation. In the next chapter we shall see that the second approximation will produce in r or x, the terms $A_1me'\cos\phi'$, $A_2ma/a'\cos\xi$, (A_1, A_2 numerical coefficients). On proceeding to a third approximation we should substitute the results of the second approximation in, for instance, x^2. We thus get amongst others a term of the form $A_3m^2e'(a/a')\cos(\xi+\phi')$, that is, in the equations for finding the *third* approximation we have a term of the same order as that in the disturbing function and therefore of the same order as that which would be used to find the *second* approximation. Hence, as far as this term is concerned, it is necessary, not only for the *form* of the solution to be correct, but also that the method of continued approximation may be applicable, to include certain parts of the equations which, in the second approximation, would ordinarily be neglected.

There are other terms for which the third approximation appears to produce coefficients of the same order as those given by the second approximation; this peculiarity is chiefly due to the direct and indirect effects of small divisors. De Pontécoulant* treats them by leaving the coefficients indeterminate until the higher approximations have been completed. Such terms illustrate the necessity, mentioned in Chap. IV. and insisted on here, of continually bearing in mind the effects produced by the higher approximations and the impossibility of obtaining the first and second approximations in the correct form, without considering them.

119. There are certain results which we cannot stop to prove here but the statement of which may perhaps prevent misconceptions. They are:—(α) The coefficients resulting from the action of the Sun in the coordinates are never of an order lower than the second; (β) The coefficients can always be represented by series of *positive* powers of m, e, e', γ, a/a', with numerical coefficients; (γ) The characteristic of a term in radius vector or longitude is the same as that of the term in R from which it arose: in latitude, it is always less by one power of γ; (δ) Nearly, but not quite, all the terms in the coordinates arising from the action of the Sun, have the factor m in its first power at least; (ϵ) the constant portions of the expansions of the functions considered contain only even powers of e, e', γ, a/a'.

* *Système du Monde*, Vol. IV. pp. 103, 145, 151, etc. The terms in longitude and radius vector of the form considered above, are those numbered 74.

With reference to the statement (γ), it may be remarked that the divisors arising from integration are, in de Pontécoulant's method, linear or quadratic functions, with integral coefficients, of n, n', cn, gn. The constants $1-c$, $1-g$ will be found to be represented by infinite series in powers of m, e^2, γ^2, e'^2, $(a/a')^2$. Their principal parts begin with the power m^2, so that the divisors involving c, g always contain powers of m. Hence none of these will have e, γ, e', a/a', as a factor.

The exceptions to (δ) as given by Delaunay*, are the terms in radius vector and longitude with arguments $D+l'+pl+2qF$, and those in latitude with arguments $D+l'+pl+(2q+1)F$, or, with the notation used here, the terms with arguments

$$\xi+\phi'+p\phi+2q\eta, \quad \xi+\phi'+p\phi+(2q+1)\eta,$$

respectively ($p, q=-\infty \ldots +\infty$).

120. Since the expression of R has the factor m^2, when we put $m=0$ all terms dependent on the action of the Sun should vanish in the expressions of the coordinates. The apparent exception of the terms just mentioned has been explained by Gogou†. It depends on the definitions of the constants in disturbed motion. When m is put zero the motions of the perigee and node vanish and the arguments of those periodic terms which remain, contain t only in the form $pnt+$const. After suitable changes of the arbitraries have been made, Delaunay's expressions with m zero reduce to those for purely elliptic motion.

On the subjects of Arts. 117—118, see de Pontécoulant, *Théorie du Système du Monde*. Vol. IV. Nos. 9—14, 90, 91, 100; Laplace, *Mécanique Céleste*, Book VII., 5. On the short period terms whose coefficients are large in comparison with their characteristics, see also Part III. of a paper by the author, *Investigations in the Lunar Theory*‡.

121. *The Second Approximation to R.*

It must be remembered that the substitution in R of elliptic values for the coordinates of the Moon, is only a means of finding a first approximation to R. Suppose that with these elliptic values for r, v, s substituted in the right-hand members of the equations (A), Art. 13, we have solved the equations and have found the new values $r+\delta r$, $v+\delta v$, $s+\delta s$, of the coordinates. According to the principles of the method, these new values must be substituted in the right-hand members of the equations in order to find the third approximation.

Let Q be a function of the coordinates of the Moon which may contain also the time. Put

$$Q=F(r, v, s),$$

$$Q+\delta Q=F(r+\delta r, v+\delta v, s+\delta s),$$

where δQ is the new part of Q arising from the additions δr, δv, δs to the elliptic values of the coordinates. Expanding by Taylor's theorem,

* *Mém. de l'Acad. des Sc.*, Vol. XXIX., Chap. XI.

† *Ann. de l'Obs. de Paris, Mém.*, Vol. XVIII. E. pp. 1—26.

‡ *Amer. Journ. Math.*, Vol. XVII. pp. 318—358.

$$\delta Q = \frac{\partial Q}{\partial r}\delta r + \frac{\partial Q}{\partial v}\delta v + \frac{\partial Q}{\partial s}\delta s$$
$$+ \frac{1}{2!}\left\{\frac{\partial^2 Q}{\partial r^2}\delta r^2 + \frac{\partial^2 Q}{\partial v^2}\delta v^2 + \frac{\partial^2 Q}{\partial s^2}\delta s^2 + 2\frac{\partial^2 Q}{\partial r \partial v}\delta r \delta v + \ldots + \ldots\right\}$$
$$+ \ldots \qquad (10).$$

By putting R, $\partial R/\partial \xi$ ... successively for Q we can find the new values of these functions. In the partials $\partial Q/\partial r$, $\partial^2 Q/\partial r^2$, etc. we substitute the initial values of the elements.

Care must be taken when we are proceeding to form such expressions as $\delta \int dt \partial R/\partial v$. Omitting, for the sake of illustration, powers of δr, δv higher than the first and all terms dependent on the latitude, we have

$$\delta \int \frac{\partial R}{\partial v} dt = \int \delta \frac{\partial R}{\partial v} dt = \int \left(\frac{\partial^2 R}{\partial v^2}\delta v + \frac{\partial^2 R}{\partial r \partial v}\delta r\right) dt.$$

This, if we regard only the terms independent of a/a', gives

$$\delta \int \frac{\partial R}{\partial v} dt = \int \left(\frac{\partial^2 R}{\partial \xi^2}\delta v + \frac{2}{r}\frac{\partial R}{\partial \xi}\delta r\right) dt \qquad (11),$$

in which we substitute for $\partial^2 R/\partial \xi^2$, $\partial R/\partial \xi$ the values obtained from (5).

(ii) *Expansion of R for Delaunay's Theory.*

122. Let α_3, α_2 be the angular distances $x\Omega$, ΩA (fig. 4, Art. 44), so that $\alpha_3 = \theta$, $\alpha_2 = \varpi - \theta$ as in Art. 84. Suppose for a moment that the Sun's orbit is inclined to the plane of xy and let α_3', α_2' be the distances $x\Omega'$, $\Omega' A'$, where A' is the solar perigee and Ω' the intersection of the Sun's orbit with the plane of xy. When the inclination of the Sun's orbit vanishes, Ω' will become an indeterminate point on xy, but $\alpha_2' + \alpha_3' = \varpi'$ will be determinate. For symmetry, we use α_2', α_3' although they can only occur in the form $\alpha_2' + \alpha_3'$.

We have from the spherical triangle $M\Omega m'$,

$$S = \cos Mm' = \cos \Omega M \cos \Omega m' + \sin \Omega M \sin \Omega m' \cos i \qquad (12),$$

or, since $\qquad \Omega M = f + \alpha_2, \quad \Omega m' = f' + \alpha_2' + \alpha_3' - \alpha_3,$

$$S = (1 - \gamma_1^2)\cos(f + \alpha_2 + \alpha_3 - f' - \alpha_2' - \alpha_3') + \gamma_1^2 \cos(f + \alpha_2 - \alpha_3 + f' + \alpha_2' + \alpha_3')$$
$$\qquad (13),$$

in which we have put $\sin \frac{1}{2} i = \gamma_1$.

From this we may form S^2, S^3 ... and, after expressing them as sums of cosines of multiples of angles, substitute them in (1).

The first part of R will consist of five separate terms of the form

$$K \frac{m' r^2}{r'^3} \cos\{2p(f + \alpha) + 2p'(f' + \alpha')\},$$

p, p' taking the pairs of values 0, 0; 0, 1; 1, 0; 1, 1; 1, −1; K being a function of γ_1^2, and α, α' depending on the angles α_2, α_3, α_2', α_3'.

Delaunay proceeds by expanding

$$\frac{r^2}{a^2},\ \frac{r^2}{a^2}\frac{\cos}{\sin}2(f+\alpha),\ \frac{a'^3}{r'^3},\ \frac{a'^3}{r'^3}\frac{\cos}{\sin}2(f'+\alpha'),$$

(where α, α' may be any angles) in powers of e, e' and cosines or sines of multiples of w, w'. These may be obtained by means of the formulæ given in Chap. III. above. By the direct multiplication of series he is then able to form all the terms required. In a similar manner the rest of the terms in R may be found.

The arguments of all the terms will evidently be composed of the four angles,

$$w,\ w',\ w+\alpha_2,\ \alpha_3-(w'+\alpha_2'+\alpha_3'),$$

or of
$$w,\ w',\ w+\alpha_2,\ w+\alpha_2+\alpha_3-(w'+\alpha_2'+\alpha_3').$$

123. It is not difficult to see that, after one or two small changes, this method of development will produce the same result as that obtained in Art. 114. In both cases we shall have expanded in terms of the mean anomalies and elements of the two orbits. In the former case the inclination of the Moon's orbit was introduced through v and s, while in Delaunay's method it is introduced directly through S. For simplicity and ease of calculation the latter method has a great advantage over the former, and moreover, it admits of a much more general treatment.

Since $\gamma=\tan i$, $\gamma_1=\sin\frac{1}{2}i$, if we put

$$\gamma_1^2=\tfrac{1}{2}-\tfrac{1}{2}(1+\gamma^2)^{-\frac{1}{2}}=\tfrac{1}{4}\gamma^2-\tfrac{3}{16}\gamma^4+\ldots,$$

and for
$$w,\ w',\ w+\alpha_2,\ w+\alpha_2+\alpha_3-(w'+\alpha_2'+\alpha_3'),$$

the symbols ϕ, ϕ', η, ξ, respectively, we shall immediately obtain the development (5).

Delaunay has performed the expansion so as to include in R all quantities up to the 8th order inclusive; in addition certain terms are carried to the 9th, 10th and even higher orders where it appears to be necessary for accuracy. His development of R consists of a constant term and 320 periodic terms. See *Mém. de l'Acad. des Sc.*, Vol. XXVIII., Chap. II.

(iii) *Hansen's development.*

124. Hansen's method is a more general one than either of those outlined above since it is adapted to the case in which the Sun's orbit is in motion. It will give, after a few small changes, the expressions both of de Pontécoulant and Delaunay.

Let ω, ω' be the angular distances of the apses of the instantaneous orbits of the Moon and the Sun from the line of intersection of the planes of the orbits, that is, from their common node, and let J be the angle between

the planes (see Arts. 217, 220). The angular distances of the two bodies from this node will be $\omega+f$, $\omega'+f'$, and therefore

$$S=\cos(f+\omega)\cos(f'+\omega')+\sin(f+\omega)\sin(f'+\omega')\cos J$$
$$=(1-\sin^2\tfrac{1}{2}J)\cos(f-f'+\omega-\omega')+\sin^2\tfrac{1}{2}J\cos(f+f'+\omega+\omega')\ \ldots(14).$$

Let $R/\mu=R^{(1)}+R^{(2)}+\ldots$, where $\mu R^{(p)}$ is the term in (1) with coefficient $m'r^{p+1}/r'^{p+2}$. Put $m'a^3/\mu a'^3=m_1^2$. Then

$$aR^{(1)}=m_1^2\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3(\tfrac{3}{2}S^2-\tfrac{1}{2})$$
$$=m_1^2\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3\{\beta_1+\beta_2\cos(2f-2f'+2\omega-2\omega')+\beta_3\cos(2f+2\omega)$$
$$+\beta_4\cos(2f'+2\omega')+\beta_5\cos(2f+2f'+2\omega+2\omega')\}\ldots(15),$$

where $\beta_1\ldots\beta_5$ are definite functions of $\sin^2\tfrac{1}{2}J$ which it is not necessary to specify here. It is required to replace r/a, a'/r', f, f' by series involving the mean anomalies w, w' and the eccentricities e, e'.

The symbol m_1 is here used instead of m because Hansen puts $m'+\mu=n'^2a'^3$ and so does not neglect the small ratio $\mu : m'$. We have then

$$m_1^2=\frac{m'a^3}{\mu a'^3}=\frac{(m'+\mu)a^3}{\mu a'^3}\Big/\left(1+\frac{\mu}{m'}\right)=\frac{n'^2}{n^2}\Big/\left(1+\frac{\mu}{m'}\right)=m^2\Big/\left(1+\frac{\mu}{m'}\right).$$

The ratio of the difference between m_1 and m to either, is the very small quantity 1/660,000. See Art. 53.

125. Let $\quad \dfrac{r^2}{a^2}=\Sigma P_j\cos jw,\quad \dfrac{a'^3}{r'^3}=\Sigma K_{j'}\cos j'w',$

$$\frac{r^2}{a^2}\begin{matrix}\cos\\ \sin\end{matrix}(2f)=\Sigma\begin{matrix}Q_j^c\\ Q_j^s\end{matrix}\begin{matrix}\cos\\ \sin\end{matrix}(jw),\quad \frac{a'^3}{r'^3}\begin{matrix}\cos\\ \sin\end{matrix}(2f')=\Sigma\begin{matrix}G_{j'}^c\\ G_{j'}^s\end{matrix}\begin{matrix}\cos\\ \sin\end{matrix}(j'w'),$$

in which j, j' receive all integral values from $+\infty$ to $-\infty$ *. The coefficients P, Q will be functions of e only and K, G functions of e' only; these may be calculated after the methods explained in Chap. III. Since, in each case, the coefficients for positive or for negative values of j, j' are superfluous, it is supposed that

$$P_{-j},\ K_{-j'},\ Q_{-j}^c,\ G_{-j'}^c,\ Q_{-j}^s,\ G_{-j'}^s=P_j,\ K_{j'},\ Q_j^c,\ G_{j'}^c,\ -Q_j^s,\ -G_{j'}^s\ldots(16),$$
respectively.

Consider trigonometrical series of the forms

$$\Sigma E_j\begin{matrix}\cos\\ \sin\end{matrix}(jw),\quad \Sigma E'_{j'}\begin{matrix}\cos\\ \sin\end{matrix}(j'w'),$$

* The letters c, s placed above the coefficients are simply marks to distinguish between the coefficients of the cosine and sine.

and of the same nature as those just given. Their products may be expressed in the forms,

$$\Sigma E_j \cos(jw) \times \Sigma E_{j'}' \,{}^{\cos}_{\sin}(j'w') = \Sigma\Sigma E_j E_{j'}' \,{}^{\cos}_{\sin}(jw + j'w'),$$

$$\Sigma E_j \sin jw \times \Sigma E_{j'}' \sin j'w' = -\Sigma\Sigma E_j E_{j'}' \cos(jw + j'w').$$

Applying these results to the term of $R^{(1)}$ with coefficient β_1 we obtain,

$$\left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 = \Sigma P_j K_{j'} \cos(jw + j'w').$$

Also for the term with coefficient β_2,

$$\begin{aligned}
&\left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \cos(2f - 2f' + 2\omega - 2\omega') \\
&= \cos(2\omega - 2\omega')\left(\frac{r^2}{a^2}\cos 2f \,.\, \frac{a'^3}{r'^3}\cos 2f' + \frac{r^2}{a^2}\sin 2f \,.\, \frac{a'^3}{r'^3}\sin 2f'\right) \\
&\qquad + \sin(2\omega - 2\omega')\left(\frac{r^2}{a^2}\cos 2f \,.\, \frac{a'^3}{r'^3}\sin 2f' - \frac{r^2}{a^2}\sin 2f \,.\, \frac{a'^3}{r'^3}\cos 2f'\right) \\
&= \cos(2\omega - 2\omega')\,\Sigma\Sigma\,(Q_j^c G_{j'}^c - Q_j^s G_{j'}^s)\cos(jw + j'w') \\
&\qquad + \sin(2\omega - 2\omega')\,\Sigma\Sigma\,(Q_j^c G_{j'}^s - Q_j^s G_{j'}^c)\sin(jw + j'w') \\
&= \tfrac{1}{2}\Sigma\Sigma\,(Q_j^c G_{j'}^c - Q_j^s G_{j'}^s - Q_j^c G_{j'}^s + Q_j^s G_{j'}^c)\cos(jw + j'w' + 2\omega - 2\omega') \\
&\qquad + \tfrac{1}{2}\Sigma\Sigma\,(Q_j^c G_{j'}^c - Q_j^s G_{j'}^s + Q_j^c G_{j'}^s - Q_j^s G_{j'}^c)\cos(jw + j'w' - 2\omega + 2\omega').
\end{aligned}$$

Since j, j' receive negative as well as positive values, we can in the second line of this expression put $-j$, $-j'$ for j, j'. Whence from the relations (16) the expansion becomes

$$\Sigma\Sigma\,(Q_j^c + Q_j^s)(G_{j'}^c - G_{j'}^s)\cos(jw + j'w' + 2\omega - 2\omega').$$

This form of the product of two series is a sufficiently simple one to calculate, when we have obtained the values of Q in terms of e and of G in terms of e'. We can express by similar formulæ the terms in $R^{(1)}$ whose coefficients are β_3, β_4, β_5.

The terms in $R^{(2)}$ are treated in like manner. The terms in $R^{(3)}$, being at least of the order $m_1^2 a^2/a'^2$, need not be calculated by means of the general formulæ. It is sufficient to choose out the various terms which will have sensible coefficients and to find the latter directly.

126. Hansen's expansions are given in the *Fundamenta*, pp. 159—179. His papers, *Entwicklung des Products einer Potenz des Radius etc.* and *Entwicklung der negativen und ungraden Potenzen der Quadratwurzel der Function* $r^2 + r'^2 - 2rr'(\cos U \cos U' + \sin U \sin U' \cos J)$*,

* *Abhandl. d. K. Sächs. Ges. d. Wissensch.*, Vol. IV. pp. 181—281, 283—376.

contain methods for the complete development of the disturbing function both in the lunar and the planetary theories. A very clear and concise account of Hansen's method and results has been given by Cayley in his first Memoir *On the Development of the Disturbing Function in the Lunar Theory**. Reference may also be made to two other papers by the same writer on the development of the disturbing function†.

In order to deduce de Pontécoulant's developments from those of Hansen we put $J=i$ and ϕ, ϕ', η, ξ for w, w', $w+\omega$, $w+\omega-w'-\omega'$ respectively. To deduce Delaunay's results we put $\sin\frac{1}{2}J=\gamma_1$, also a_2 for ω, and $a_2'+a_3'-a_3$ for ω'.

(iv) *Laplace's Equations.*

127. In order to develope F for the purpose of treating Laplace's equations (Chap. II.), we have by Arts. 8, 107,

$$F=\frac{\mu}{r}+\frac{m'r^2}{r'^3}\left(\tfrac{3}{2}S^2-\tfrac{1}{2}\right)+\frac{m'r^3}{r'^4}\left(\tfrac{5}{2}S^3-\tfrac{3}{2}S\right)+\ldots.$$

And since

$$\frac{1}{r}=\frac{1}{r_1\sqrt{(1+s^2)}}=\frac{u_1}{\sqrt{(1+s^2)}},\quad r'=\frac{1}{u'},\quad S=\frac{\cos(v-v')}{\sqrt{(1+s^2)}},$$

we obtain
$$F=\frac{\mu u_1}{\sqrt{(1+s^2)}}+\frac{m'u'^3}{u_1^2}\left\{\tfrac{3}{2}\cos^2(v-v')-\tfrac{1}{2}(1+s^2)\right\}+\ldots\ldots\ldots\ldots(17).$$

Laplace forms the forces $\partial F/\partial u_1$, $\partial F/\partial v$, $\partial F/\partial s$ directly from this expression. Since the independent variable is v, it is then necessary to expand the results in terms of v and of the elements.

By means of the results of Art. 52 the coordinates u_1, s are immediately put into the required form. The coordinates u', v' being given as functions of t must be expressed by means of the results of Arts. 52, 70 in terms of v. See Laplace, *Méc. Cél.*, Book VII. Chap. I.

(v) *Equations referred to Rectangular Coordinates.*

128. The expansion of the disturbing function for the equations of Arts. 18—20, has been there performed as far as it is necessary. It is a feature of the method that we do not substitute elliptic values for the coordinates of the Moon in the terms dependent on the action of the Sun.

The part Ω_2 of equation (16), Chap. II., is that portion of Ω which is independent of the parallax of the Sun and which vanishes when e' is zero. As already pointed out in Art. 22, we can put

$$\Omega_2=\mathrm{A}v^2+2\mathrm{B}v\sigma+\mathrm{C}\sigma^2+\mathrm{K}z^2\ldots\ldots\ldots\ldots(18),$$

in which A, B, C, K depend only on the motion of the Sun and are at least of the order m^2e'; they can be easily expanded in powers of e' by the known elliptic formulæ.

* *Mem. of R. Astr. Soc.*, Vol. XXVII. (1859). *Coll. Works*, Vol. III. pp. 293—318.

† *Mem. R. Astr. Soc.*, Vols. XXVIII., XXIX. *Coll. Works*, Vol. III. pp. 319—343, 360—474.

CHAPTER VII.

DE PONTÉCOULANT'S METHOD.

129. We have, in Section (i), Chapter II., obtained the equations (A) on which de Pontécoulant has based his method. In Chapter III. Art. 50, are to be found the elliptic values of the coordinates which serve as a first approximation after the modification, formulated in Chapter IV. Art. 68, has been made. In Chapter VI. Art. 114, we have given a development of R obtained by using these modified elliptic values; and, in the same division of Chapter VI., certain theorems which tend to simplify the algebraical processes of the second and higher approximations, are proved. The object of this chapter is to explain the manner of carrying out the various approximations, by applying the principles already discussed to the discovery of some of the larger inequalities in the Moon's motion. The arrangement of the inequalities into classes, although a natural one, is not essential to the method (Art. 151). It enables us, however, to explain the origin of the various periodic terms in the expressions for the coordinates and to carry out portions of the third approximation with greater ease and security.

Preparation of the equations (A) *for the second and higher approximations.*

130. Let r_0, u_0 be the modified elliptic values of r, $u\,(=1/r)$. Let

$$1/r = 1/r_0 + \delta u,$$

and therefore $$r^2 = r_0^2 - 2r_0^3\,\delta u + 3\,(r_0^2\,\delta u)^2 - \ldots;$$

δu is then the part of $1/r$ depending directly on the disturbing action of the Sun.

Substituting for r on the left-hand side of the first of equations (A), Art. 13, we obtain, since μ is put equal to unity,

$$\tfrac{1}{2}\frac{d^2}{dt^2}r_0^2 - \frac{1}{r_0} + \frac{1}{a} - \frac{d^2}{dt^2}(r_0^3\,\delta u) - \delta u = -\tfrac{3}{2}\frac{d^2}{dt^2}(r_0^2\,\delta u)^2 + \ldots + P \ldots (1),$$

where $$P = r\frac{\partial R}{\partial r} + 2\int d'R \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since R contains only even powers of γ, the equations for radius vector and longitude—and therefore those coordinates—contain only even powers of γ. We shall neglect powers of γ beyond the first and consequently, in the first two of equations (A), neglect γ, s entirely.

We can obtain other forms for P by means of the results of Arts. 115, 116. Neglecting s we have, in the second approximation,

$$P=(p+2)R+2(p+2)\int R\frac{dr}{r}+2\int\frac{\partial R}{\partial \xi}dv \quad\text{..............}\quad (3),$$

where $p=0$ for the terms independent of the solar parallax and $p=1$ for the terms dependent on the first power of the ratio a/a'.

Also from equations (9) of Art. 116, in the second approximation,

$$P=(p+4)R+2(p+3)\int R\frac{dr'}{r'}+2\int\frac{\partial R}{\partial \xi}dv' \quad\text{...........}\quad (4),$$

with the same definition of p. An arbitrary constant is considered to be present in the expression for P. The value of δP, necessary for the approximations beyond the second, is found by Taylor's theorem as in Art. 121.

131. Let v_0 be the elliptic* value of the longitude and let h_0 be the elliptic value of h. Put

$$v=v_0+\delta v, \quad h=h_0+\delta h.$$

Neglecting s^2 and substituting, the second of equations (A) may be written,

$$\frac{dv_0}{dt}-\frac{h_0}{r_0^2}+\frac{d}{dt}\delta v=\left(h_0+\delta h+\int\frac{\partial R}{\partial v}dt\right)\left(\frac{1}{r_0^2}+\frac{2}{r_0}\delta u+(\delta u)^2\right)-\frac{h_0}{r_0^2}\text{......}(5).$$

We have also, since γ is neglected and $\mu=1$,

$$h_0=\sqrt{a(1-e^2)},$$

or, when e^2 is neglected,

$$h_0=\sqrt{a}=na^2 \text{..............................}(6).$$

Finally, the equation to be used to find the latitude is not the third of equations (A) but

$$\ddot{z}+\frac{z}{r^3}=\frac{\partial R}{\partial z}=\frac{1}{r}\frac{\partial R}{\partial s}+\frac{s}{r}r\frac{\partial R}{\partial r} \quad\text{......................}\quad (7),$$

from Art. 13, neglecting s^3.

The equation (6) of Chapter II. for the determination of the constant part of $1/r$ is, when we neglect γ^2, a/a',

$$\frac{\ddot{r}}{r}-\dot{v}^2+\frac{1}{r^3}=\frac{2R}{r^2} \quad\text{............................}\quad (8).$$

* That is, modified elliptic. This abbreviation will be used throughout the chapter.

When we are considering terms independent of e, we have

$$\frac{1}{2}\frac{d^2}{dt^2}r_0^2 - \frac{1}{r_0} + \frac{1}{a} = 0, \quad \frac{dv_0}{dt} - \frac{h_0}{r_0^2} = 0,$$

and not otherwise. For the introduction of c will cause terms of the order $1 - c$ to appear if we substitute the *modified* elliptic values of r_0, v_0 in these equations. They are only satisfied by the *purely* elliptic values of r_0, v_0.

The constant δh and that considered to be present in P are theoretically superfluous, but the presence of δh is of great assistance in determining the meanings to be attached to the arbitraries in disturbed motion.

132. We have seen in Chapter VI. that the characteristic of a term in R is unaltered by the integration of the radius vector and longitude equations. All terms in R depending on the latitude are at least of the order γ^2: when introduced into the latitude equation they will be at least of the order γ. The order of the characteristic is not further lowered by the integration of this equation.

Hence we can divide up the terms of the disturbing function and, instead of finding the complete first approximation with all the terms of R, we can separate them out according to the composition of their characteristics.

The order in which the terms will be taken is as follows: the terms whose coefficients depend only on (i) m; (ii) m, e; (iii) m, e'; (iv) $m, a/a'$; (v) m, γ; (vi) m and any combinations of $e, e', \gamma, a/a'$ and of their powers. In the second, third, fourth and fifth classes we shall here develope only the terms depending on the *first* powers of $e, e', a/a', \gamma$ respectively; the terms in the sixth class will not be developed.

The approximations, which will in certain cases be carried to the third order*, are made according to powers of the disturbing forces, that is, of m^2. In the first approximation we neglected the disturbing forces; in the second approximation all the new parts added should be at least of the order m^2; in the third approximation of the order m^4, and so on. But, owing to the small divisors introduced by integration, as already explained in the previous chapter, some of the terms in the second and third approximations contain m in its first power. It is this fact which causes the great labour necessary to produce expressions for the coordinates with an accuracy comparable with that of the best lunar observations of the present day.

133. It is necessary to make mention here of the two constants n, a. In undisturbed motion we have

$$n^2a^3 = 1.$$

In disturbed motion, n will be defined (Art. 135) as the observed mean motion; n, a are two of the arbitraries of the solution. Since we cannot have seven independent arbitraries, *the relation $n^2a^3 = 1$ will be supposed to*

* The details of the third approximation are printed in small type.

hold between the symbols n, a in disturbed motion, whatever may be the meaning attached to n. When n has been defined, a definite meaning is thereby given to a.

The necessary changes in the meanings to be attached to n, e, γ when the motion is disturbed, are defined in the course of the chapter. Fuller explanations will be given in Chapter VIII.

(i) *The terms whose coefficients depend only on m.*

134. From equation (5), Art. 114 we have, since all terms dependent on e, e', γ, a/a' are neglected,

$$R = \frac{m^2}{a}\left(\tfrac{1}{4} + \tfrac{3}{4}\cos 2\xi\right) \qquad\qquad (9);$$

$$r_0 = a, \; v_0 = nt + \epsilon, \qquad r' = a', \; v' = n't + \epsilon'.$$

Also, as $2\xi = 2(n - n')t + 2\epsilon - 2\epsilon'$, we see by the results of Art. 117, that none of the terms here considered will have the orders of their coefficients lowered by integration. Hence, in the results of the second approximation all terms will be of the order m^2 at least, in those of the third approximation of the order m^4, and so on.

In the second approximation we neglect powers of δu higher than the first. The equation (1) therefore gives, since $n^2a^3 = 1$,

$$-\frac{1}{n^2}\frac{d^2}{dt^2}\delta u - \delta u = P.$$

And from equation (4), since we neglect a/a' and therefore have $p = 0$, we obtain

$$P = 4R + 2n'\int \frac{\partial R}{\partial \xi}\,dt.$$

From the value of R given above,

$$\frac{\partial R}{\partial \xi} = -\tfrac{3}{2}\frac{m^2}{a}\sin 2\xi;$$

therefore $$2n'\int \frac{\partial R}{\partial \xi}\,dt = \tfrac{3}{2}\frac{m^2}{a}\frac{n'}{n - n'}\cos 2\xi.$$

The equation for δu then becomes, since $n'/n = m$,

$$-\frac{1}{n^2}\frac{d^2}{dt^2}\delta u - \delta u = \frac{m^2}{a}(1 + 3\cos 2\xi) + \tfrac{3}{2}\frac{m^2}{a}\frac{m}{1 - m}\cos 2\xi + \frac{\alpha}{a},$$

where α/a is the constant attached to the integral in P.

To find the particular integral assume

$$\delta u = \frac{1}{a}(b_0 + b_2\cos 2\xi).$$

Substituting and equating the constant term and the coefficient of $\cos 2\xi$ to zero, we have

$$-b_0 = m^2 + \alpha \qquad (10),$$

$$+4b_2\left(\frac{n-n'}{n}\right)^2 - b_2 = 3m^2 + \tfrac{3}{2}\frac{m^3}{1-m}.$$

The second of these gives

$$b_2\{4(1-m)^2 - 1\} = 3m^2 + \tfrac{3}{2}m^3/(1-m),$$

or, expanding in powers of m,

$$b_2 = m^2 + \tfrac{19}{6}m^3 + \tfrac{137}{18}m^4 + \ldots \qquad (11).$$

The third approximation will produce terms of the order m^4 and this value of b_2 is therefore only correct to m^3; it is given here to the order m^4 in preparation for the next approximation. Since α is an arbitrary constant, the constant term b_0 is at present undetermined.

135. The equation (5) for the longitude is, with the same substitutions and to the same degree of approximation, since $h_0 = na^2$,

$$\frac{d}{dt}\delta v = 2na\,\delta u + \frac{1}{a^2}\int \frac{\partial R}{\partial \xi}dt + \frac{\delta h}{a^2}.$$

This, from the values of δu, $\int \frac{\partial R}{\partial \xi}dt$ found above, becomes as far as the order m^3,

$$\frac{d}{dt}\delta v = 2n\{b_0 + (m^2 + \tfrac{19}{6}m^3)\cos 2\xi\} + \tfrac{3}{4}\frac{m^2n^2}{n-n'}\cos 2\xi + \frac{\delta h}{a^2}.$$

Therefore integrating,

$$\delta v + B = \left(2nb_0 + \frac{\delta h}{a^2}\right)t + \frac{n}{n-n'}\left(m^2 + \tfrac{19}{6}m^3 + \tfrac{3}{8}m^2\frac{n}{n-n'}\right)\sin 2\xi$$

$$= \left(2nb_0 + \frac{\delta h}{a^2}\right)t + (\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \ldots)\sin 2\xi \qquad (12),$$

where we have put $n/(n-n') = (1-m)^{-1} = 1 + m + m^2 + \ldots$.

The longitude is $v_0 + \delta v = nt + \epsilon + \delta v$. This shows that the introduction of B is useless, for it merely adds an arbitrary part to ϵ which was itself arbitrary.

The coefficient of t is $n + 2nb_0 + \delta h/a^2$. Since n was an arbitrary of the original solution and since δh is an arbitrary introduced into the second approximation we can determine the latter at will. We shall always give this arbitrary a value such that *the coefficient of t in the expression for the true longitude is always denoted by n*; this statement defines the meaning of n in disturbed motion. Thus $2nb_0 + \delta h/a^2 = 0$ and

$$\delta v = (\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3)\sin 2\xi \qquad (13),$$

as far as the order m^3.

136. The constant term b_0 in δu is found by substituting the values $1/a+\delta u$, $nt+\epsilon+\delta v$ for $1/r$, v in (8). That is to say, we put

$$\dot{v}=n+(n-n')(\tfrac{11}{4}m^2+\tfrac{59}{6}m^3)\cos 2\xi,\quad \frac{a}{r}=1+b_0+(m^2+\tfrac{19}{6}m^3)\cos 2\xi;$$

therefore, as b_0 is of the order m^2 at least,

$$\dot{v}^2=n^2+n(n-n')(\tfrac{11}{2}m^2+\tfrac{59}{3}m^3)\cos 2\xi,\ \frac{a^3}{r^3}=1+3b_0+(3m^2+\tfrac{19}{2}m^3)\cos 2\xi;$$

and similarly for r/a, $\ddot{r}/a$.

The equation gives, since $2R/a^2$ can be put for $2R/r^2$,

$$\{1+b_0+(m^2+\tfrac{19}{6}m^3)\cos 2\xi\}\,4(n-n')^2(m^2+\tfrac{19}{6}m^3)\cos 2\xi$$
$$-n^2-n(n-n')(\tfrac{11}{2}m^2+\tfrac{59}{3}m^3)\cos 2\xi$$
$$+\frac{1}{a^3}\{1+3b_0+(3m^2+\tfrac{19}{2}m^3)\cos 2\xi\}=\frac{m^2}{a^3}(\tfrac{1}{2}+\tfrac{3}{2}\cos 2\xi).$$

Put $n-n'=n(1-m)$, $1/a^3=n^2$ and expand in powers of m, neglecting those beyond m^3. On equating the coefficient of $\cos 2\xi$ and the constant term to zero, it will be found that the former vanishes identically, thus giving a verification of the previous work, while the latter furnishes to the order m^3,

$$-n^2+n^2(1+3b_0)=\tfrac{1}{2}m^2n^2;$$

or, correctly to the order m^3, there being no term of that order,

$$b_0=\tfrac{1}{6}m^2 \quad\text{.................................}\quad (14).$$

Thence
$$\delta u=\frac{1}{a}\{\tfrac{1}{6}m^2+(m^2+\tfrac{19}{6}m^3)\cos 2\xi\}\quad\text{.................}\quad (15),$$

to the order m^3.

137. In order to exhibit the method of finding the third approximation, we shall calculate the coefficients of the periodic terms in $1/r$ up to the order m^4. They can be found to the order m^5 in this approximation, but for the purpose of explanation it is not necessary to include the terms of this order as they merely involve further expansions.

To find the third approximation we must include the first term of the right-hand member of (1); this term being of the order m^4 at the lowest, we can use the result (15) of the second approximation. We have therefore

$$-\tfrac{3}{2}\frac{d^2}{dt^2}(r_0{}^2\delta u)^2=-\tfrac{3}{2}\frac{d^2}{dt^2}\left\{a^4\frac{m^4}{a^2}(\tfrac{1}{6}+\cos 2\xi)^2\right\}\qquad \text{to } m^4,$$

$$=2\frac{m^4}{a}\cos 2\xi+12\frac{m^4}{a}\cos 4\xi,\qquad ,,$$

since $(n-n')^2m^4=n^2m^4=m^4/a^3$ to the order m^4.

Also from (4),

$$\delta P = 4\delta R + 2n' \int \delta \frac{\partial R}{\partial v} dt$$

$$= 4 \frac{\partial R}{\partial r} \delta r + 4 \frac{\partial R}{\partial v} \delta v + 2n' \int \left(\frac{\partial^2 R}{\partial r \partial v} \delta r + \frac{\partial^2 R}{\partial v^2} \delta v \right) dt$$

$$= \frac{8}{r} R \delta r + 4 \frac{\partial R}{\partial \xi} \delta v + 2n' \int \left(\frac{2}{r} \frac{\partial R}{\partial \xi} \delta r + \frac{\partial^2 R}{\partial \xi^2} \delta v \right) dt.$$

The terms being all of the order m^4 at least, we can here put $r = a$. Substituting the values of R, $\partial R/\partial \xi$, $\partial^2 R/\partial \xi^2$ obtained from (9) and those of δv and $\delta r = -r^2 \delta u = -a^2 \delta u$ from (13) and (15), we obtain to the order m^4,

$$\delta P = -8 \frac{m^2}{a} (\tfrac{1}{4} + \tfrac{3}{4} \cos 2\xi)(\tfrac{1}{6} m^2 + m^2 \cos 2\xi) - 4 \frac{m^2}{a} \tfrac{3}{2} \sin 2\xi \, (\tfrac{11}{8} m^2 \sin 2\xi)$$

$$+ 2n' \int \left\{ 2 \frac{m^2}{a} \tfrac{3}{2} \sin 2\xi \, (\tfrac{1}{6} m^2 + m^2 \cos 2\xi) - 3 \frac{m^2}{a} \cos 2\xi \, (\tfrac{11}{8} m^2 \sin 2\xi) \right\} dt \; \ldots (16).$$

On integration, the second line of this expression will be seen to be of the order m^5. Hence

$$\delta P = -\frac{m^4}{a} (\tfrac{179}{24} + 3 \cos 2\xi - \tfrac{9}{8} \cos 4\xi) \qquad \text{to } m^4.$$

Let the new part of $1/r$ be $\delta_1 u$. The equation for $\delta_1 u$ therefore becomes

$$-\frac{1}{n^2} \frac{d^2}{dt^2} \delta_1 u - \delta_1 u = -\frac{m^4}{a} \cos 2\xi + \tfrac{105}{8} \frac{m^4}{a} \cos 4\xi + \text{const. part.}$$

We do not put the constant terms into evidence since, as in the first approximation, they are determined by means of (8).

Assuming $$\delta_1 u = \frac{1}{a} (\delta b_0 + \delta b_2 \cos 2\xi + b_4 \cos 4\xi),$$

we easily obtain $$\delta_1 u = \frac{1}{a} (\delta b_0 - \tfrac{1}{3} m^4 \cos 2\xi + \tfrac{7}{8} m^4 \cos 4\xi).$$

Adding this to the value of δu which in (11) was carried to the order m^4 for the coefficient of $\cos 2\xi$, we have

$$\delta u + \delta_1 u = \frac{1}{a} \{\tfrac{1}{6} m^2 + \delta b_0 + (m^2 + \tfrac{19}{6} m^3 + \tfrac{131}{18} m^4) \cos 2\xi + \tfrac{7}{8} m^4 \cos 4\xi\}.$$

138. In a similar manner the third approximation to the longitude may be calculated. In doing this we can omit all the constant terms which appear in the equation for dv/dt, since these merely add a known part and an arbitrary part to the coefficient of t in the expression for v, and these new parts, by the definition of n, always vanish. After the value of v has been found we can obtain the constant part of $1/r$ by means of (8) and at the same time verify the results previously obtained.

When these calculations have been performed we shall get as far as m^4 for the terms whose coefficients depend on m only,

$$\left. \begin{aligned} \frac{a}{r} &= 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (m^2 + \tfrac{19}{6} m^3 + \tfrac{131}{18} m^4) \cos 2\xi + \tfrac{7}{8} m^4 \cos 4\xi, \\ v &= nt + \epsilon \qquad + (\tfrac{11}{8} m^2 + \tfrac{59}{12} m^3 + \tfrac{893}{72} m^4) \sin 2\xi + \tfrac{201}{256} m^4 \sin 4\xi, \end{aligned} \right\} \ldots (17).$$

From the remarks just made, it is evident that *in calculating the right-hand member of the radius equation we can always omit the constant portions.* This evidently applies to the calculation of all inequalities.

Also, *in calculating the terms in the equation for $\dot{v}$, we always equate the constant portions to zero.* This rule also applies to all inequalities.

(ii) *The terms whose coefficients depend on m and the first power of e only.*

139. The part of R required is*

$$R = \frac{m^2}{a}\{\tfrac{1}{4} + \tfrac{3}{4}\cos 2\xi - \tfrac{1}{2}e\cos\phi - \tfrac{9}{4}e\cos(2\xi - \phi) + \tfrac{3}{4}e\cos(2\xi + \phi)\} \ldots (18).$$

Also, from Art. 50, since γ, e^2 are neglected,

$$r_0 = a(1 - e\cos\phi), \quad v_0 = nt + \epsilon + 2e\sin\phi, \quad s = 0,$$
$$r' = a', \quad v' = n't + \epsilon'.$$

We first neglect all powers of m beyond m^2. In forming P for the second approximation, it is evident that we do not need the first two terms of R. Hence by (4), remembering that $\xi = (n - n')t + \epsilon - \epsilon'$, $\phi = cnt + \epsilon - \varpi$,

$$P = \frac{m^2}{a}e\{-2\cos\phi - 9\cos(2\xi - \phi) + 3\cos(2\xi + \phi)\}$$
$$+ \frac{2m^2n'}{a}e\left\{-\tfrac{9}{4}\frac{2\cos(2\xi - \phi)}{2n - 2n' - cn} + \tfrac{3}{4}\frac{2\cos(2\xi + \phi)}{2n - 2n' + cn}\right\} \;\ldots\ldots (19).$$

Since $1 - c$ is of the order m at least, the second line of this expression is of the order m^3 and therefore, to the order m^2, the value of P is given by the first line.

The particular integral will be

$$a\delta u = \tfrac{1}{6}m^2 + m^2\cos 2\xi + ec_0\cos\phi + ec_{-1}\cos(2\xi - \phi) + ec_1\cos(2\xi + \phi).$$

The first two terms are those obtained in (i); c_0, c_{-1}, c_1 are the coefficients to be found. We must substitute this expression for δu in (1).

We have $$\frac{d^2}{dt^2}(r_0^3\delta u) = \frac{d^2}{dt^2}\{a^3(1 - 3e\cos\phi)\delta u\}.$$

Whence, retaining only terms with the characteristic e,

$$\frac{d^2}{dt^2}(r_0^3\delta u) = a^2e\frac{d^2}{dt^2}\{-\tfrac{1}{2}m^2\cos\phi - 3m^2\cos 2\xi\cos\phi + c_0\cos\phi$$
$$+ c_{-1}\cos(2\xi - \phi) + c_1\cos(2\xi + \phi)\}$$
$$= a^2e\frac{d^2}{dt^2}\{(c_0 - \tfrac{1}{2}m^2)\cos\phi + (c_{-1} - \tfrac{3}{2}m^2)\cos(2\xi - \phi) + (c_1 - \tfrac{3}{2}m^2)\cos(2\xi + \phi)\}.$$

* It is necessary to include the first two terms since, in combination with other terms of the order e, they may produce coefficients of the order considered. There is the same necessity for all inequalities.

Substituting this and the value (19) of P in equation (1) and neglecting all terms but those which have the first power of e in their coefficients, we obtain

$$c^2n^2a^2e\cos\phi - \frac{e}{a}\cos\phi - \frac{e}{a}\{c_0\cos\phi + c_{-1}\cos(2\xi-\phi) + c_1\cos(2\xi+\phi)\}$$
$$+ n^2a^2e\,\{(c_0-\tfrac{1}{2}m^2)\,c^2\cos\phi + (c_{-1}-\tfrac{3}{2}m^2)(2-2m-c)^2\cos(2\xi-\phi)$$
$$+ (c_1-\tfrac{3}{2}m^2)(2-2m+c)^2\cos(2\xi+\phi)\}$$
$$= \frac{m^2}{a}\,e\,\{-2\cos\phi - 9\cos(2\xi-\phi) + 3\cos(2\xi+\phi)\}.$$

Putting $1/a$ for n^2a^2 and equating the coefficients of $\cos\phi$, $\cos(2\xi-\phi)$, $\cos(2\xi+\phi)$ to zero we have, after multiplication by a/e, the three equations of condition

$$c^2 - 1 - c_0 + c^2(c_0 - \tfrac{1}{2}m^2) = -2m^2,$$
$$-c_{-1} + (c_{-1} - \tfrac{3}{2}m^2)(2-2m-c)^2 = -9m^2,$$
$$-c_1 \;\; + (c_1 \;\; - \tfrac{3}{2}m^2)(2-2m+c)^2 = +3m^2.$$

The first of these may be written

$$(c^2-1)(1+c_0) = -2m^2 + \tfrac{1}{2}m^2c^2 \quad\text{.....................(20).}$$

As c_0, $c-1$ are known to be of the order m at least, this shows that $c-1$ is of the order m^2 at least. Hence, neglecting all powers of m beyond the second,

$$c^2 - 1 = -2m^2 + \tfrac{1}{2}m^2 = -\tfrac{3}{2}m^2,$$

or

$$c = 1 - \tfrac{3}{4}m^2$$

to the order m^2. This value agrees with that found in Art. 67.

The other two equations of condition then give, neglecting all powers of m but the lowest present,

$$c_{-1} = \tfrac{15}{8}m, \quad c_1 = \tfrac{33}{16}m^2\,;$$

showing that the coefficient of $\cos(2\xi-\phi)$ has been lowered one order by the integration—a fact which might have been predicted, since the coefficient of t in $2\xi-\phi$ differs from n by a quantity of the first order.

It is evident that c_0 is a new arbitrary constant, for the term $c_0\cos\phi$ might be considered to be included in the elliptic value of a/r. We shall not, however, neglect it here, but leave it arbitrary until the longitude has been found.

140. To calculate the corresponding terms in longitude, the equation (5) gives to the order required, namely to m^2e,

$$\frac{dv_0}{dt} - \frac{h_0}{r_0^2} + \frac{d}{dt}\,\delta v = \frac{2h_0}{r_0}\,\delta u + \frac{\delta h}{r_0^2} + \frac{1}{r_0^2}\int\frac{\partial R}{\partial\xi}\,dt.$$

Substituting the values of the various terms we have, since here $h_0 = na^2$,

$$2(c-1)\,ne\cos\phi + \frac{d}{dt}\,\delta v = 2n\,(1+e\cos\phi)\,\{\tfrac{1}{6}m^2 + m^2\cos 2\xi + ec_0\cos\phi$$
$$+\tfrac{15}{8}me\cos(2\xi-\phi) + \tfrac{33}{16}m^2e\cos(2\xi+\phi)\}$$
$$+\frac{1}{a^2}\,(1+2e\cos\phi)\left[\delta h + \frac{m^2}{a}\left\{\tfrac{3}{4}\,\frac{2\cos 2\xi}{2\,(n-n')} - \tfrac{9}{4}\,\frac{2e\cos(2\xi-\phi)}{2n-2n'-cn} + \tfrac{3}{4}\,\frac{2e\cos(2\xi+\phi)}{2n-2n'+cn}\right\}\right].$$

After expanding these expressions, we omit those periodic terms independent of e; we also put $1/a^3 = n^2$ and $c=1$ in those coefficients which are of the orders em or em^2. We then have

$$\frac{1}{n}\frac{d}{dt}\,\delta v = \tfrac{1}{3}m^2 + \frac{\delta h}{na^2} + e\left(-2c+2+2c_0+\tfrac{1}{3}m^2+2\,\frac{\delta h}{na^2}\right)\cos\phi$$
$$+\tfrac{15}{4}me\cos(2\xi-\phi) + \tfrac{51}{8}m^2e\cos(2\xi+\phi),$$

in which terms of a higher order than those required are neglected.

The constant term is to be put zero. Hence

$$\delta h = -\tfrac{1}{3}na^2m^2.$$

Since c_0 is arbitrary we can determine it at will. Let it be such that *the coefficient of* $\sin\phi$ *in longitude is the same as in elliptic motion.* This gives the definition of e in disturbed motion.

We have therefore,

$$\frac{1}{c}\left(-2c+2+2c_0+\tfrac{1}{3}m^2+2\,\frac{\delta h}{na^2}\right) = 0 \quad\text{..................(21)},$$

or, giving to c, δh their values,

$$c_0 = -\tfrac{7}{12}m^2.$$

Integrating the longitude equation, we finally obtain

$$\left.\begin{aligned} a\delta u &= -\tfrac{7}{12}m^2e\cos\phi + \tfrac{15}{8}me\cos(2\xi-\phi) + \tfrac{33}{16}m^2e\cos(2\xi+\phi),\\ \delta v &= \qquad\qquad\qquad \tfrac{15}{4}me\sin(2\xi-\phi) + \tfrac{17}{8}m^2e\sin(2\xi+\phi),\end{aligned}\right\}\text{...(22)},$$

in which the terms with argument $2\xi-\phi$ are correct to the order me and those with arguments ϕ, $2\xi+\phi$ to the order m^2e.

141. We can, by paying attention to the orders of the terms, obtain c to the order m^3 without much further labour. Its value was obtained by equating the coefficient of $\cos\phi$ in the radius-vector equation to zero. The result (20) may be written

$$c^2-1+c_0(c^2-1) = -\tfrac{3}{2}m^2+\tfrac{1}{2}m^2(c^2-1)\text{..........................(23)}.$$

It was seen that c^2-1, c_0 are of the order m^2 at least, so that $c_0(c^2-1)$, $\tfrac{1}{2}m^2(c^2-1)$ are of the order m^4 at least. It is not then necessary to further approximate to c_0. The new portions of the coefficient of $\cos\phi$ in (1) can therefore only arise from the terms

$$-\tfrac{3}{2}\frac{d^2}{dt^2}(r_0{}^2\delta u)^2+\delta P = -\tfrac{3}{2}\frac{d^2}{dt^2}(r_0{}^2\delta u)^2+4\delta R+2n'\int\delta\,\frac{\partial R}{\partial\xi}\,dt\text{............(24)}.$$

Now all terms in R are of the order m^2 at least, while only those terms in δu, δv with argument $2\xi-\phi$ are of the order me. Hence a term of the order m^3e, with the argument ϕ, can only arise in the part δP of (24) from the term in R of argument 2ξ combined with the term in δu or δv of argument $2\xi-\phi$. The last term of the expression (24) furnishes only portions of the order m^4e, and it can therefore be neglected. As (Art. 121)

$$\delta R=-2Rr_0\,\delta u+\frac{\partial R}{\partial \xi}\,\delta v,$$

we can, in this equation, put

$$r_0=a,\quad R=\frac{m^2}{a}\,\tfrac{3}{4}\cos 2\xi,\quad \delta u=\tfrac{15}{8}\frac{me}{a}\cos(2\xi-\phi),\quad \delta v=\tfrac{15}{4}me\sin(2\xi-\phi).$$

Whence $\quad 4\delta R=-\tfrac{45}{4}\dfrac{m^3e}{a}\cos 2\xi\cos(2\xi-\phi)-\tfrac{45}{2}\dfrac{m^3e}{a}\sin 2\xi\sin(2\xi-\phi),$

of which the part depending on the argument ϕ gives

$$\delta P=4\delta R=-\tfrac{135}{8}\frac{m^3e}{a}\cos\phi.$$

In the term $-\frac{3}{2}d^2(r_0{}^2\delta u)^2/dt^2$ we can, by similar reasoning, put

$$r_0=a,\quad a\delta u=m^2\cos 2\xi+\tfrac{15}{8}me\cos(2\xi-\phi)\,;$$

whence $\qquad (r_0{}^2\delta u)^2$ contributes $\tfrac{15}{8}m^3a^2e\cos\phi,$

and $\qquad -\tfrac{3}{2}\dfrac{d^2}{dt^2}(r_0{}^2\delta u)^2$ contributes $\tfrac{3}{2}\,\tfrac{15}{8}m^3a^2ec^2n^2\cos\phi=\tfrac{45}{16}\dfrac{m^3e}{a}\cos\phi,$

to the order required.

Adding this to the value of δP, the new term of argument ϕ in the right-hand member of (1) is

$$-\tfrac{225}{16}\frac{m^3e}{a}\cos\phi.$$

The equation of condition (23) therefore becomes, neglecting quantities of the order m^4,

$$c^2-1=-\tfrac{3}{2}m^2-\tfrac{225}{16}m^3\,;$$

or, taking the square root,

$$c=1-\tfrac{3}{4}m^2-\tfrac{225}{32}m^3\quad\text{.......................... (25).}$$

(iii) *The terms whose coefficients depend on m and the first power of e′ only.*

142. These are very easy to calculate and we shall only indicate the steps. There are no indeterminate coefficients to find as in cases (i), (ii). We put

$$R=\frac{m^2}{a}\left\{\tfrac{1}{4}+\tfrac{3}{4}\cos 2\xi+\tfrac{3}{4}e'\cos\phi'+\tfrac{21}{8}e'\cos(2\xi-\phi')-\tfrac{3}{8}e'\cos(2\xi+\phi')\right\},$$

$$r_0=a,\quad v_0=nt+\epsilon,\quad s=0.$$

We use here the equation (3) to calculate P. The equation (1) becomes

$$-\frac{1}{n^2}\frac{d^2}{dt^2}\delta u-\delta u=\frac{m^2}{a}e'\left\{\tfrac{3}{2}\cos\phi'+\tfrac{21}{2}\cos(2\xi-\phi')-\tfrac{3}{2}\cos(2\xi+\phi')\right\},$$

giving, as far as the order m^2e',

$$a\delta u=-\tfrac{3}{2}m^2e'\cos\phi'+\tfrac{7}{2}m^2e'\cos(2\xi-\phi')-\tfrac{1}{2}m^2e'\cos(2\xi+\phi')\ldots(26).$$

Thence we obtain

$$\frac{d}{dt}\delta v = 2na\,\delta u + \frac{m^2 e'}{a^3}\left\{\tfrac{21}{8}\frac{2\cos(2\xi-\phi')}{2n-2n'-n'} - \tfrac{3}{8}\frac{2\cos(2\xi+\phi')}{2n-2n'+n'}\right\}$$

$$= nm^2e'\{-3\cos\phi' + \tfrac{77}{8}\cos(2\xi-\phi') - \tfrac{11}{8}\cos(2\xi+\phi')\},$$

and by integration

$$\delta v = -3me'\sin\phi' + \tfrac{77}{16}m^2e'\sin(2\xi-\phi') - \tfrac{11}{16}m^2e'\sin(2\xi+\phi')\ \ldots(27).$$

One term in δv has been lowered to the order me' by integration. This will cause terms of the order m^3e' to appear in the third approximation to $1/r$ and therefore, we should suppose, terms of the order m^2e' in the third approximation to v. But the only new terms in the radius-equation and in $\int dt\,\delta(\partial R/\partial\xi)$ which are of the order m^3e' will be easily seen to be those with arguments $2\xi \pm \phi'$ and no terms are lowered in order by the integration of the radius-equation; hence, the only new terms which are of order m^3e' in the longitude-equation are those with arguments $2\xi \pm \phi'$. Therefore, as the only terms in the longitude-equation which can have their orders lowered by integration are those of argument ϕ' and as such terms are at least of the order m^4e', *the values* (26), (27) *for* δu, δv *are correct to the order* m^2e'.

The third approximation may be carried out as in cases (i), (ii).

(iv) *The terms whose coefficients depend on* m *and the first power of* a/a' *only.*

143. For these terms, we have

$$R = \frac{m^2}{a}\left\{\tfrac{1}{4} + \tfrac{3}{4}\cos 2\xi + \tfrac{3}{8}\frac{a}{a'}\cos\xi + \tfrac{5}{8}\frac{a}{a'}\cos 3\xi\right\},$$

$$r_0 = a,\quad v_0 = nt+\epsilon,\quad s=0,\quad r'=a',\quad v'=n't+\epsilon',$$

and from (4),

$$P = (p+4)R + 2n'\int\frac{\partial R}{\partial\xi}\,dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28).$$

In the second approximation to $1/r$ we shall not require the first two terms of R: $p=1$ for the third and fourth terms of R. Since the coefficient of t in the argument of ξ differs from n by a quantity of the order mn, the coefficient of $\cos\xi$ will be lowered one order by the integration of the radius equation. We shall therefore, in preparation for the third approximation, carry this term to the order m^3a/a' in the radius equation so that, after the third approximation is complete, we may have it correct to the order m^2a/a'. The calculations are very similar to those necessary for case (iii).

We therefore have, to the order m^3a/a' in the coefficient of $\cos\xi$ and to the order m^2a/a' in that of $\cos 3\xi$,

$$P = 5R + \frac{2n'}{n-n'}\frac{m^2}{a}\tfrac{3}{8}\frac{a}{a'}\cos\xi = 5R + \tfrac{3}{4}\frac{m^3}{a}\frac{a}{a'}\cos\xi;$$

and equation (1) becomes,

$$-\frac{1}{n^2}\frac{d^2}{dt^2}\delta u - \delta u = \frac{m^2}{a}\frac{a}{a'}\left(\tfrac{15}{8}+\tfrac{3}{4}m\right)\cos\xi + \tfrac{25}{8}\frac{m^2}{a}\frac{a}{a'}\cos 3\xi.$$

The particular integral is

$$a\delta u = \frac{a}{a'}\left(\tfrac{15}{8}m^2+\tfrac{3}{4}m^3\right)\frac{n^2\cos\xi}{(n-n')^2-n^2} + \tfrac{25}{8}m^2\frac{a}{a'}\frac{n^2\cos 3\xi}{9(n-n')^2-n^2}$$

$$= -\left(\tfrac{15}{16}m+\tfrac{27}{32}m^2\right)\frac{a}{a'}\cos\xi + \tfrac{25}{64}m^2\frac{a}{a'}\cos 3\xi,$$

in which the coefficient of $\cos\xi$ is only correct to the order ma/a'.

The equation for the longitude becomes, after neglecting superfluous terms,

$$\frac{d}{dt}\delta v = 2\frac{h_0}{a}\delta u + \frac{1}{a^2}\int\frac{\partial R}{\partial\xi}dt$$

$$= n\frac{a}{a'}\left\{-\left(\tfrac{15}{8}m+\tfrac{27}{16}m^2\right)\cos\xi + \tfrac{25}{32}m^2\cos 3\xi\right\}$$

$$+\frac{m^2}{a^3}\frac{a}{a'}\frac{1}{n-n'}\left\{\tfrac{3}{8}\cos\xi+\tfrac{5}{8}\cos 3\xi\right\}$$

$$= n\frac{a}{a'}\left\{-\left(\tfrac{15}{8}m+\tfrac{21}{16}m^2\right)\cos\xi + \tfrac{45}{32}m^2\cos 3\xi\right\},$$

giving

$$\delta v = -\left(\tfrac{15}{8}m+\tfrac{51}{16}m^2\right)\frac{a}{a'}\sin\xi + \tfrac{15}{32}m^2\frac{a}{a'}\sin 3\xi,$$

where the coefficient of $\sin\xi$ is only correct to the order ma/a'.

144. *The third approximation.* We are only going to find δu, δv correctly to the order m^2a/a'. In order to do so, we require the new terms of order m^2a/a' in $1/r$, that is, the new terms of order m^3a/a' and of argument ξ in equation (1). The only way in which such terms can be produced is from the term in δu or in δv of argument ξ, order ma/a', combined with terms of arguments 0, 2ξ, order m^2.

We have from (28), since the second term, being multiplied by n', may be omitted,

$$\delta P = 4\delta R \text{ or } 5\delta R,$$

and

$$\delta R = \frac{\partial R}{\partial\xi}\delta v - 2Rr_0\delta u \text{ or } \frac{\partial R}{\partial\xi}\delta v - 3Rr_0\delta u,$$

according as we take terms in R independent of a/a' or dependent on its first power. The only way in which we can get a term of argument ξ, order m^3a/a', from this expression is by putting

$$R = \frac{m^2}{a}\left(\tfrac{1}{4}+\tfrac{3}{4}\cos 2\xi\right),\quad a\delta u = -\tfrac{15}{16}m\frac{a}{a'}\cos\xi,\quad \delta v = -\tfrac{15}{8}m\frac{a}{a'}\sin\xi.$$

Whence, as the terms considered in R are independent of a/a',

$$\delta R = \tfrac{45}{16}\frac{m^3}{a}\frac{a}{a'}\sin\xi\sin 2\xi + \tfrac{15}{32}\frac{m^3}{a}\frac{a}{a'}\cos\xi + \tfrac{45}{32}\frac{m^3}{a}\frac{a}{a'}\cos\xi\cos 2\xi\,;$$

and therefore the terms of argument ξ in δP are given by

$$\delta P = \tfrac{165}{16}\frac{m^3}{a}\frac{a}{a'}\cos\xi.$$

For the first term on the right-hand side of (1) we take

$$a\delta u = \tfrac{1}{8}m^2 + m^2\cos 2\xi - \tfrac{15}{16}m\frac{a}{a'}\cos\xi;$$

therefore the term of argument ξ in $(a\delta u)^2$ is

$$-\tfrac{5}{16}m^3\frac{a}{a'}\cos\xi - \tfrac{15}{16}m^3\frac{a}{a'}\cos\xi = -\tfrac{5}{4}m^3\frac{a}{a'}\cos\xi;$$

so that $$-\tfrac{3}{2}\frac{d^2}{dt^2}(r_0^2\delta u)^2 \text{ contributes } -\tfrac{15}{8}\frac{m^3}{a}\frac{a}{a'}\cos\xi.$$

Let the new part of $1/r$ be $\delta_1 u$. We then have

$$-\frac{1}{n^2}\frac{d^2}{dt^2}\delta_1 u - \delta_1 u = (\tfrac{165}{16} - \tfrac{15}{8})\frac{m^3}{a}\frac{a}{a'}\cos\xi = \tfrac{135}{16}\frac{m^3}{a}\frac{a}{a'}\cos\xi.$$

Integrating $$\delta_1 u = \tfrac{135}{16}\frac{m^3}{a}\frac{a}{a'}\frac{n^2\cos\xi}{(n-n')^2 - n^2} = -\tfrac{135}{32}\frac{m^2}{a}\frac{a}{a'}\cos\xi.$$

Adding this to the value of δu previously found we obtain,

$$a\delta u = -(\tfrac{15}{16}m + \tfrac{81}{16}m^2)\frac{a}{a'}\cos\xi + \tfrac{25}{64}m^2\frac{a}{a'}\cos 3\xi, \quad \text{correct to } m^2\frac{a}{a'} \quad \ldots (29).$$

We shall now obtain the corresponding term in longitude. Since no coefficient has its order lowered by the integration, the new portion is simply that arising from the term $\delta_1 u$ just found and it is of the order $m^2 a/a'$. Let it be denoted by $\delta_1 v$.

In the equation (5) the new portion $\delta\int dt\,\partial R/\partial\xi$ is at least of the order $m^3 a/a'$ and so contributes nothing. The same is true of $(\delta u)^2$, $\delta h \times \delta u$, $\delta u \times \int dt\,\partial R/\partial\xi$.

The only new portion of the order $m^2 a/a'$ is therefore given by

$$\frac{d}{dt}\delta_1 v = \frac{2h}{r_0}\delta_1 u = -\tfrac{135}{16}nm^2\frac{a}{a'}\cos\xi,$$

whence $$\delta_1 v = -\tfrac{135}{16}\frac{n}{n-n'}m^2\frac{a}{a'}\sin\xi = -\tfrac{135}{16}m^2\frac{a}{a'}\sin\xi.$$

Adding this to the value of δv previously found, we have

$$\delta v = -(\tfrac{15}{8}m + \tfrac{93}{8}m^2)\frac{a}{a'}\sin\xi + \tfrac{15}{32}m^2\frac{a}{a'}\sin 3\xi, \quad \text{correct to } m^2\frac{a}{a'} \ldots (30).$$

(v) *The terms whose coefficients depend on m and the first power of γ.*

145. We have $s = \text{tang. of lat.} = z\sqrt{(1+s^2)}/r$, by Art. 12.

Since the inclination of the plane of the Moon's orbit is always a small quantity oscillating about a mean value of 5°, we can consider s—and consequently z—as small quantities of the first order and, in the first instance, neglect their squares and higher powers. Also, we have seen that the radius

vector and the longitude will contain only even powers of the small quantity γ (which is always a factor of s), and that the latitude will contain it in odd powers only. Hence in the equation for s or in the equation for z we are only neglecting quantities of the order s^3 or z^3.

As stated in Art. 131, the equation (7) will be used to find z and thence to obtain the latitude. In order to express its right-hand member in terms of the coordinates, we use the development (3) of Chapter VI., neglecting the parallax of the Sun, that is, neglecting the terms beyond the first. We thus obtain,

$$\frac{\partial R}{\partial z} = -\tfrac{3}{2}\frac{m'rs}{r'^3}\{1+\cos 2(v-v')\} + \tfrac{1}{2}\frac{m'rs}{r'^3}\{1+3\cos 2(v-v')\}$$

$$= -\frac{m'rs}{r'^3} = -\frac{m'z}{r'^3} \text{ to the order required*.}$$

The equation of motion therefore takes the simple form,

$$\ddot{z} + z\left(\frac{1}{r^3} + n'^2\frac{a'^3}{r'^3}\right) = 0;$$

or, neglecting e' and dividing by n^2, the form,

$$\frac{1}{n^2}\ddot{z} + z\left(\frac{a^3}{r^3} + m^2\right) = 0,$$

an equation which is sufficient to determine all terms in latitude whose coefficients depend on m, e and the first power of γ when those portions of r which depend on m, e, are known. We shall, however, neglect e and therefore give to a/r the part of its value dependent on m only—that given by the first of equations (17) as far as the order m^4.

We are, in this method of finding the latitude, apparently departing from the principles laid down in Chapter IV. concerning the method of solution by continued approximation. That is to say, instead of considering the first or elliptic approximation to s as known and then proceeding to find the new part due to the action of the Sun, we are including both in the equation of motion, so that we shall find a portion of the first and second approximations at one step. For the purposes here we do not need the value of s given in Art. 50.

Put $a/r = 1 + a\delta u$. We have

$$a^3/r^3 = 1 + 3a\delta u + 3(a\delta u)^2 + (a\delta u)^3.$$

The value (17) of δu gives $(a\delta u)^3 = 0$ to the order m^4, and to the same order

$$3(a\delta u)^2 = 3\left(\tfrac{1}{36}m^4 + \tfrac{1}{3}m^4\cos 2\xi + m^4\cos^2 2\xi\right)$$
$$= \tfrac{19}{12}m^4 + m^4\cos 2\xi + \tfrac{3}{2}m^4\cos 4\xi.$$

The equation for z therefore becomes,

$$\frac{1}{n^2}\ddot{z} + z\left\{1 + \tfrac{3}{2}m^2 - \tfrac{9}{32}m^4 + \left(3m^2 + \tfrac{19}{2}m^3 + \tfrac{137}{6}m^4\right)\cos 2\xi + \tfrac{33}{8}m^4\cos 4\xi\right\} = 0.$$

* This result is nevertheless true when quantities of the orders γ^3, γ^5... are taken into account. The only quantity actually neglected, when $r^2 = x^2 + y^2 + z^2$, is a/a'. See Art. 150.

146. This equation is a case of the general form,

$$\ddot{z}+n^2z\{1+\Sigma_j q_j \cos 2jt\}=0, \quad (j=1, 2, \ldots \infty),$$

which frequently occurs in physical problems. It is of principal importance in celestial mechanics for the determination of the mean motions of the perigee and the node.

The solution is of the form

$$z=\Sigma_j q_j' \cos (gnt+2jt+\alpha), \quad (j=-\infty \ldots +\infty),$$

the arbitrary constants being α and one of the q_j'. The chief part of the problem, which we cannot stop here to investigate generally, is the determination of g. Hill and Adams, as we shall see in Chapter XI., find g from the equation by means of an infinite determinant. For the purposes of Case (v) we shall assume the solution to be of the form given above and find the value of g to the required order by continued approximation.

An investigation of the differential equation and of its solutions will be found, together with a large number of references to the literature on the subject, in Tisserand, *Méc. Cél.* Vol. III. Chaps. I. II.

147. Assume as the solution,

$$\frac{z}{a}=\gamma\{g_0\sin\eta+g_{-1}\sin(2\xi-\eta)+g_1\sin(2\xi+\eta)+g_{-2}\sin(4\xi-\eta)+g_2\sin(4\xi+\eta)\},$$

where γg_0 is considered to be one arbitrary constant. The symbol η stands for $gnt+\epsilon-\theta$, where θ is the other arbitrary constant: g is a constant as yet undetermined.

If we put $m=0$ the motion is undisturbed and we shall have $g=1$; also

$$z=a\gamma g_0 \sin \eta.$$

In undisturbed motion, $\gamma=\tan i$. If therefore we put $g_0=1$, we shall be able to define γ in disturbed motion as being such *that the coefficient of the principal term in z is the same as in undisturbed motion.* It is evident that, in undisturbed motion, θ, ϵ are the same as in Chapter III.

We put then $g_0=1$ and substitute the assumed value of z. It will appear, when we write down the equations of condition, that since g differs from unity by a quantity of the order m at least, g_{-1}, g_1, g_{-2}, g_2 are of the orders m, m^2, m^3, m^4 respectively. Hence, omitting terms beyond the order m^4, the equations of condition become,

$$\left.\begin{aligned}
1-g^2+\tfrac{3}{2}m^2-\tfrac{9}{32}m^4-(\tfrac{3}{2}m^2+\tfrac{19}{4}m^3)\,g_{-1}+\tfrac{3}{2}m^2\,g_1&=0\\
\{1-(2-2m-g)^2\}\,g_{-1}+\tfrac{3}{2}m^2g_{-1}-(\tfrac{3}{2}m^2+\tfrac{19}{4}m^3+\tfrac{137}{12}m^4)&=0\\
\{1-(2-2m+g)^2\}\,g_1+\tfrac{3}{2}m^2g_1+(\tfrac{3}{2}m^2+\tfrac{19}{4}m^3+\tfrac{137}{12}m^4)&=0\\
\{1-(4-4m-g)^2\}\,g_{-2}-\tfrac{33}{16}m^4+(\tfrac{3}{2}m^2+\tfrac{19}{4}m^3)\,g_{-1}&=0\\
\{1-(4-4m+g)^2\}\,g_2+\tfrac{33}{16}m^4-(\tfrac{3}{2}m^2+\tfrac{19}{4}m^3)\,g_1&=0
\end{aligned}\right\}\ (31).$$

Neglecting powers of m beyond the second, we obtain from the first of these,

$$1-g^2=-\tfrac{3}{2}m^2, \quad \text{or} \quad g=1+\tfrac{3}{4}m^2.$$

Therefore from the second and third,

$$g_{-1}(4m - m^2) = \tfrac{3}{2}m^2 + \tfrac{19}{4}m^3 \qquad \text{to } m^3,$$

$$g_1(-8 + 12m - 7m^2) = -\tfrac{3}{2}m^2 - \tfrac{19}{4}m^3 - \tfrac{137}{12}m^4 \qquad \text{to } m^4;$$

or

$$g_{-1} = \tfrac{3}{8}m + \tfrac{41}{32}m^2 \qquad \text{to } m^2,$$

$$g_1 = \tfrac{3}{16}m^2 + \tfrac{7}{8}m^3 + \tfrac{989}{384}m^4 \qquad \text{to } m^4.$$

With these values we obtain from the fourth and fifth of the equations,

$$g_{-2}(-8 + 24m) = \tfrac{33}{16}m^4 - (\tfrac{3}{2}m^2 + \tfrac{19}{4}m^3)(\tfrac{3}{8}m + \tfrac{41}{32}m^2) \qquad \text{to } m^4,$$

$$g_2\ (-24) = -\tfrac{33}{16}m^4 + \tfrac{3}{2}m^2\,\tfrac{3}{16}m^2 \qquad \text{to } m^4;$$

giving

$$g_{-2} = \tfrac{9}{128}m^3 + \tfrac{213}{512}m^4 \qquad \text{to } m^4,$$

$$g_2 = \tfrac{19}{256}m^4 \qquad \text{to } m^4.$$

Substituting the values of g_{-1}, g_1 in the first equation we obtain for the further approximation to g,

$$1 - g^2 + \tfrac{3}{2}m^2 - \tfrac{9}{32}m^4 - (\tfrac{3}{2}m^2 + \tfrac{19}{4}m^3)(\tfrac{3}{8}m + \tfrac{41}{32}m^2) + \tfrac{9}{32}m^4 = 0,$$

or

$$g^2 = 1 + \tfrac{3}{2}m^2 - \tfrac{9}{16}m^3 - \tfrac{237}{64}m^4 \qquad \text{to } m^4,$$

or

$$g = 1 + \tfrac{3}{4}m^2 - \tfrac{9}{32}m^3 - \tfrac{273}{128}m^4 \qquad \text{to } m^4 \text{ .... (32).}$$

With this value of g we can, from the second of equations (31), obtain a more approximate value of g_{-1}, namely,

$$g_{-1}(4m - m^2 - \tfrac{57}{16}m^3) = \tfrac{3}{2}m^2 + \tfrac{19}{4}m^3 + \tfrac{137}{12}m^4 \qquad \text{to } m^4,$$

or

$$g_{-1} = \tfrac{3}{8}m + \tfrac{41}{32}m^2 + \tfrac{5389}{1536}m^3 \qquad \text{to } m^3.$$

The value of z is therefore obtained to the order γm^4 in all coefficients except in that of $\sin(2\xi - \eta)$ which is found to the order γm^3. As far as the order γm^2, we have,

$$\frac{z}{a} = \gamma \sin \eta + (\tfrac{3}{8}m + \tfrac{41}{32}m^2)\gamma \sin(2\xi - \eta) + \tfrac{3}{16}m^2\gamma \sin(2\xi + \eta) \text{...(33).}$$

Also to this order, from the first of equations (17),

$$s = z/r = z(1 + \tfrac{1}{6}m^2 + m^2 \cos 2\xi)/a.$$

Whence, to the order γm^2,

$$s = (1 + \tfrac{1}{6}m^2)\gamma \sin \eta + (\tfrac{3}{8}m + \tfrac{25}{32}m^2)\gamma \sin(2\xi - \eta) + \tfrac{11}{16}m^2\gamma \sin(2\xi + \eta) \text{... (34).}$$

(vi) *The terms whose coefficients are dependent on m and the products and higher powers of e, e′, γ, a/a′.*

148. It is not intended to develope the algebraical expressions of the coordinates for terms other than those just given. To find the terms included in this class the developments become of great length, but there is

no change in the general method of finding them. When we wish to find any particular inequality defined either by its argument or by the order of its coefficient relative to $e, e', \gamma, a/a'$, it is in general sufficient for the first approximation to the terms in $1/r$, to choose the corresponding terms present in R; to find the terms in longitude and latitude, we must consider those of lower characteristics in $1/r$, s (these are supposed to have been previously found) and in R which, by their combinations, may produce terms of the required order. We shall then, as far as powers of m are concerned, have a first approximation to the term. It is generally necessary to proceed to further approximations.

If, after the third approximation to the coordinates has been completed, the order of the new portion in the coefficient of any particular term is not higher than that of the portion with the same characteristic obtained in the second approximation, we must, in computing the second approximation, leave the corresponding coefficient indeterminate until the third approximation is reached: the coefficient may then be found. In this case the second approximation is not capable of giving the first term of the series for the coefficient in powers of m. See Arts. 118–120.

Summary of the results.

149. We shall now collect all the results of the first and second approximations and those portions of the third which have been determined in the previous articles. The numerical coefficient of each term in the coefficients is correct to the order given. The elliptic parts of the values of the coordinates will not be written down; they are to be found in Art. 50.

The various portions of a/r are given by equations (17), (22), (26), (29). Whence,

$$\begin{aligned}\frac{a}{r} = \text{Elliptic value} &+ \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4 + (m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4)\cos 2\xi + \tfrac{7}{8}m^4 \cos 4\xi \\ &- \tfrac{7}{12}m^2 e \cos\phi + \tfrac{15}{8}me\cos(2\xi - \phi) + \tfrac{33}{16}m^2 e \cos(2\xi + \phi) \\ &- \tfrac{3}{2}m^2 e' \cos\phi' + \tfrac{7}{2}m^2 e' \cos(2\xi - \phi') - \tfrac{1}{2}m^2 e' \cos(2\xi + \phi') \\ &- (\tfrac{15}{16}m + \tfrac{81}{16}m^2)\frac{a}{a'}\cos\xi + \tfrac{25}{64}m^2 \frac{a}{a'}\cos 3\xi.\end{aligned}$$

The various portions of v given in equations (17), (22), (27), (30), furnish,

$$\begin{aligned}v = \text{Elliptic value} &+ (\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4)\sin 2\xi + \tfrac{201}{256}m^4 \sin 4\xi \\ &+ \tfrac{15}{4}me\sin(2\xi - \phi) + \tfrac{17}{8}m^2 e \sin(2\xi + \phi) \\ &- (3m + 0m^2)e'\sin\phi' + \tfrac{77}{16}m^2 e' \sin(2\xi - \phi') - \tfrac{11}{16}m^2 e' \sin(2\xi + \phi') \\ &- (\tfrac{15}{8}m + \tfrac{93}{8}m^2)\frac{a}{a'}\sin\xi + \tfrac{15}{32}m^2 \frac{a}{a'}\sin 3\xi.\end{aligned}$$

The parts of these which were found by the third approximation are the terms of order m^4 and those of order $m^2 \frac{a}{a'}$ in the coefficients of $\frac{\cos}{\sin} \xi$; all the other coefficients were found from the second approximation.

The value of s, as given by (34), furnishes,

$$s = \text{Elliptic value} + \tfrac{1}{6}m^2\gamma \sin \eta + (\tfrac{3}{8}m + \tfrac{25}{32}m^2)\gamma \sin (2\xi - \eta) + \tfrac{11}{16}m^2\gamma \sin (2\xi + \eta).$$

We have also from equations (25), (32),

$$c = 1 - \tfrac{3}{4}m^2 - \tfrac{225}{32}m^3,$$

$$g = 1 + \tfrac{3}{4}m^2 - \tfrac{9}{32}m^3 - \tfrac{273}{128}m^4,$$

the former being only obtained correctly to the order m^3 while the latter is found correctly to the order m^4.

Finally, n is the coefficient of t in the non-periodic part of v; e, γ are such that the coefficient of $\sin \phi$ in longitude and that of $\sin \eta$ in z are the same as in undisturbed elliptic motion.

150. For the cases (i) to (v) we have simply chosen out of the development of R, given in Art. 114, the terms required. It is easy and often useful to deduce each particular case directly from the disturbing function.

We have
$$R = \frac{m'}{\sqrt{r^2 + r'^2 - 2rr'S}} - \frac{m'rS}{r'^3},$$

where S is the cosine of the angle between the radii to the Sun and the Moon.

Cases (i), (iv). Here, $e=0$, $e'=0$, $\gamma=0$; therefore $r=a$, $r'=a'$, $v=nt+\epsilon$, $v'=n't+\epsilon'$ $S=\cos(v-v')=\cos\xi$. Whence

$$R = \frac{n'^2a'^3}{\sqrt{(a^2+a'^2-2aa'\cos\xi)}} - n'^2a\cos\xi \qquad (35).$$

Expanding as in Art. 107 and putting $n'^2a^2 = m^2/a$,

$$R = \frac{m^2}{a}(\tfrac{1}{4} + \tfrac{3}{4}\cos 2\xi) + \frac{m^2}{a}\frac{a}{a'}(\tfrac{3}{8}\cos\xi + \tfrac{5}{8}\cos 3\xi) + \ldots$$

Case (v). Here $r^2 = x^2+y^2+z^2$, $S = xx'+yy'$, and therefore

$$\frac{\partial R}{\partial z} = \frac{m'z}{(r^2+r'^2-2rr'S)^{\frac{3}{2}}}.$$

If we neglect the ratio $a : a'$ or $r : r'$ this gives

$$\frac{\partial R}{\partial z} = \frac{m'z}{r'^3} = n'^2 z\left(\frac{a'}{r'}\right)^3,$$

furnishing *all* inequalities in latitude independent of the solar parallax.

Cases (ii), (iii). Here $\gamma=0$, $a/a'=0$. We obtain by expanding R,

$$R = \frac{m^2}{a}\left(\frac{a'}{r'}\right)^3\left(\frac{r}{a}\right)^2\{\tfrac{1}{4} + \tfrac{3}{4}\cos(v-v')\}.$$

For case (ii), we have $r'=a'$, $v'=n't+\epsilon'$ and r, v take their elliptic values.

For case (iii), we have $r=a$, $v=nt+\epsilon$ and r', v' take their elliptic values.

By proceeding in this way we can without much trouble obtain any class of inequality defined by its characteristic.

De Pontécoulant's method as contained in his Système du Monde, Vol. IV.

151. We have already mentioned in Art. 129 that the plan used here of dividing up the various terms into classes defined by the characteristic, is not essential for the development of the method nor is it used by de Pontécoulant. Further, if a complete development of the expressions were required, it would hardly be a saving of labour to proceed in this way. It will be readily seen that after R has been found in terms of the time to the required degree of accuracy by using the elliptic values of the radius vector, the longitude and the latitude, we should, in order to obtain the complete second approximation, use the complete elliptic value of R in the equation (1) which serves to find δu. Certain coefficients would be indeterminate and they would be left so until, with this value of δu, we had found the value of δv from equation (5), when they would be determined as in cases (i), (ii). De Pontécoulant does not give full details of the method of procedure he adopted, but it is not difficult to see his general plan which is somewhat as follows.

The first step is to neglect the latitude and with it all terms in the radius-vector equation, the longitude equation and the disturbing function, which depend on γ or s. In order to obtain a second approximation to $1/r$, v of the terms independent of γ—such terms forming by far the greater part of R—the parts on the right-hand side of (1) dependent on $(\delta u)^2$, $(\delta u)^3$... are neglected, the value of P* is calculated from the expression for R and the complete elliptic value of r_0 is substituted on the left-hand side of the equation. In order to solve the resulting equation for δu, we assume a value for this quantity which consists of a constant term and periodic terms of the same arguments as those occurring in the equation—all the coefficients being indeterminate. When this value for δu is substituted, by equating the coefficients of the various terms to zero we obtain definite values for all the indeterminate coefficients except for that of $\cos\phi$ and for the constant term. The coefficients of $\cos\phi$, however, give a first approximation to the value of c. Leaving these two coefficients indeterminate we proceed with the value of δu so obtained to find that of δv from (5). We then determine the new arbitraries after the manner explained in cases (i), (ii) and these, together with the coefficient of $\cos\phi$, become definite. The constant part of $1/r$ is found from an equation corresponding to that numbered (8) above.

With these values of δu, δv we proceed to a third approximation by finding δR according to the method of Art. 121 and we thence obtain the whole of the new terms on the right-hand side of (1). The resulting equations for the new parts of $1/r$, v are solved as before and a third approximation to these coordinates is deduced. Proceeding in this way by successive stages, all the coefficients are ultimately obtained accurately to quantities of the fifth order inclusive. In addition, certain coefficients which are either expressed by slowly converging series of powers of m, e, etc. or which, owing to the presence of small divisors, have their orders raised, are calculated to higher orders by choosing out the particular combinations required to obtain them.

The latitude equation is then treated. Neglecting powers of γ (and therefore of z and s) higher than the first and using the values of $1/r$, v already found, we can obtain z by means

* De Pontécoulant uses the letter P to denote the terms on the right-hand side of (1) which depend on $(\delta u)^2$, $(\delta u)^3$....

of equation (7); in doing so, since all terms in z have the factor γ, it will only be necessary, except for certain coefficients, to use the previously found values of $1/r$, v as far as the fourth order, if we merely require z accurately to quantities of the fifth order. When z has been obtained the value of s is easily deduced. The terms of the order γ^2 in $1/r$, v can then be calculated by methods quite similar to those used before to approximate to these coordinates. Returning to the latitude equation we find the portions of z and thence those of s which are of the order γ^3; and from the new parts of s so found we can obtain the parts in $1/r$, v which are of the order γ^4. De Pontécoulant, neglecting quantities above the fifth order, stops at this point; the terms of order γ^5 in s are simply those given by the elliptic formulæ. In the course of the approximations, the value of g is found and the meaning of γ defined in disturbed motion according to the principles explained in (v). When all these operations have been performed we shall have found complete expressions for the coordinates, accurately to the fifth order at least, as far as the action of the Sun is concerned.

152. Two important differences between the expressions obtained above and those found by de Pontécoulant, must be noted: they refer to the meanings of e, γ in disturbed motion. We have defined them to be such that the coefficient of $\sin\phi$ in longitude and that of $\sin\eta$ in z shall be the same as in undisturbed motion. De Pontécoulant, having before him the earlier results obtained by those who followed Laplace's method and desiring to compare his expressions with theirs, so determined these constants that, in the final expressions of the coordinates in terms of the time, the results, if correctly worked out, should agree. This point will be further explained in Art. 159 below.

153. The successive approximations are not given in detail by de Pontécoulant. He merely states that the labour of performing them was very great, and then proceeds to write out the complete value of R obtained by substituting the results of the various approximations in the expression given in Art. 121 above; he gives also the value of δu furnished by the previous approximations. With these expressions for R, δu, he goes on to find the complete value of $1/r$ and thence that of v, finally obtaining the value of s. The labour of performing this last approximation is divided into several portions:—first, we are given those portions independent of γ, whose coefficients include only the powers of e, e', a/a' contained in their characteristics; secondly, those terms in the latitude whose coefficients depend only on the first power of γ and on those powers of e, e', a/a' present in their characteristics; thirdly, the omitted portions of the coefficients in all the three coordinates are found.

He then proceeds to find the coefficients of certain long-period inequalities more accurately; and also to obtain the inequalities due to the actions of the planets, the non-spherical shape of the earth, etc. The results are worked out algebraically all through and the final expressions for the parallax, the longitude and the latitude are given in Chapter VII. of the volume. Numerical results are then obtained by substituting the numerical values of m, e, e', γ, a/a', $1/a$, in the coefficients, for the particular case of the Earth's Moon. These are followed by tables comparing his results with those obtained by Plana and by Damoiseau and with those obtained directly from observation.

154. A brief examination of the literal expressions for the coordinates given by de Pontécoulant in Chap. VII. of his volume will show that the series for certain coefficients, when arranged according to powers of m, appear to converge very slowly owing to the large numerical multipliers; it is doubtful whether all these series are really convergent even for the small value which m has in the case of the Moon. If we arrange them according to powers of e^2, e'^2, γ^2, $(a/a')^2$, such slow convergence does not seem to take place on account of the very small values of these four constants. It becomes important

then to consider whether we can improve the slow convergence by substituting another parameter instead of m and also how we may estimate the remainders of such series on the assumption that they do really converge (see Art. 69).

One of the series which converges very slowly is that which represents the coefficient of the term with argument $2\xi - \phi + \phi'$ and characteristic ee'. The portion of this coefficient in longitude, of the form $ee'f(m)$, as given by de Pontécoulant, is*

$$ee'\left(-\tfrac{15}{4}m - \tfrac{173}{32}m^3 + \tfrac{49045}{384}m^3 + \tfrac{29183005}{18432}m^4 + \tfrac{2827212637}{221184}m^5 + \tfrac{470658148655}{5308416}m^6\right).$$

The last term calculated has therefore a numerical multiplier of nearly 90,000 and its ratio to the first term is about $24{,}000\,m^5 = 1/18$ approximately. The numerical value of the terms given is about 30″, so that six terms of the series are not sufficient to give the number of seconds accurately, although the complete coefficient is small. It is such series as these which make the literal development tedious and difficult.

It has been shewn by Hill† that if we expand in powers of $\mathrm{m} = m/(1-m)$ instead of in powers of m, most of the series will be rendered more convergent. Further, a careful inspection will often enable us to estimate the remainders with some exactness, owing to a certain regularity which these series appear to display‡.

* *Sys. du Monde*, Vol. IV. p. 577. It must be remembered that the definition of e is that used by de Pontécoulant; this coefficient will therefore not be the same as that obtained by Delaunay. See Art. 159 below.

† "Researches in the Lunar Theory," *Amer. Journ. Math.* Vol. I. p. 141.

‡ See two notes by the author in the *Monthly Not. R. A. S.* Vol. LII. pp. 71—80; LV. pp. 3—5. In the latter, the complete numerical value of the above term is given.

CHAPTER VIII.

THE CONSTANTS AND THEIR INTERPRETATION.

155. ONE of the most important departments of the lunar theory is the interpretation of the arbitrary constants which arise when the equations of motion are integrated. It has already been mentioned that in obtaining expressions to represent the coordinates at any time, we must keep in view the necessity of putting the expressions into forms which enable us to readily give a physical interpretation of the results. In one direction—the elimination of all terms which increase in proportion with the time—this object has been achieved: the difficulty was only of an analytical nature. Another question—the convergency of the series obtained—we have been obliged to leave aside owing to the lack of any certain knowledge on the subject. Further, assuming the convergency of the series which are obtained when the problem has been formally solved and when the coordinates have been expressed in terms of the time and of certain constants, it is necessary to so determine these constants that the initial conditions of the problem may be satisfied. In many physical problems it is sufficient to know the initial coordinates and velocities in order to determine the constants easily. But the peculiar nature of the problems of celestial mechanics makes it impossible to find them with any approach to accuracy in this way; this is owing partly to the difficulty of measuring distances, partly to the inaccuracy of a single observation and partly to the complicated nature of the equations which it would be necessary to solve. Consequently, indirect methods must be resorted to.

There are three questions to be considered in the interpretation of the results. In the first place, we must give definite physical meanings to all the constants involved in order that we may be able to apply the results to the case of any satellite which moves under the general conditions initially assumed. Secondly, we must be able to determine the numerical values of the constants from observation, as accurately as the observations permit, in the case of any such satellite—and more particularly of the Moon—so

that tables may be formed which will give the place of the satellite at any time, with an error not exceeding that of a single observation. The discovery of new causes of disturbance, and of their magnitude, is rendered far easier when tables, calculated from theory alone and including the effects of all known causes of disturbance, have been formed. It is the small differences between the tables and the observations which will be most likely to lead to an advance in our knowledge of the peculiarities in the motions of the bodies forming the solar system—provided these differences cannot be wholly accounted for either by errors of observation or by inaccurate values of the constants used in forming the tables. Thirdly, when the constants have been determined, the magnitude of the effects produced by the various terms present in the expressions for the coordinates, will be inquired into.

The Signification of the Constants.

156. We have, in the previous Chapter, formed certain developments for the coordinates and, in so doing, we have introduced new arbitrary constants or defined those introduced into the first approximation in a new way. It is immediately obvious that as soon as we begin to use the results of the second approximation, the constants can no longer have their former significations. They were specially defined for the case of elliptic motion, that is, motion in a curve of known properties. When this orbit was modified by the introduction of c and g, it was still possible to interpret the results geometrically, namely, by the use of an elliptic orbit of fixed size and shape, with its apse and its node moving with uniform velocities in given directions (Art. 68). When, however, we go further and approximate to the path of the Moon by the methods of Chap. VII., no such easy interpretation of the results is possible: the curve described is not one with whose properties we are familiar. In order to use the results, it will be necessary to consider the constants separately; we must also give them such meanings that the determination of them by observation shall be as easy as possible and that the results of any other method, in which the arbitrary constants of the solution may be introduced in a different manner, can be compared with those just obtained.

Besides the arbitrary constants of solution there are present certain constants which have been supposed known, namely, those referring to the elliptic solar orbit. These are m', n', a', e', ϵ', ϖ', of which the first three are connected by the relation $m' = n'^2a'^3$. When the orbit of the Sun is supposed not to be elliptic and not to be in one plane, two new constants depending on the position of the plane of the orbit will be introduced, and all the constants must be defined again. The problem of the determination of the solar constants, although it does not differ much from that of the lunar constants, belongs mainly to the planetary theory and we shall therefore

leave it aside. In the following we retain the former suppositions—that the orbit of the Sun is elliptic and that it lies in the fixed plane of reference.

In attaching meanings to the arbitrary constants used in disturbed motion, the principal object which will be kept in view is the consideration of definitions which depend in no way on the method of integration adopted. We shall thus be able to compare the results obtained by any of the methods used for treating the lunar theory.

157. In the first approximation we had seven arbitrary constants, a, n, e, ϵ, ϖ, θ, i (or γ) and one constant μ present in the differential equations: μ was eliminated from the results by means of the necessary relation $\mu = n^2a^3$. This relation is not used to eliminate a or n because μ is a much more difficult constant to determine observationally than either a or n: μ is, in fact, considered as an unknown, although definite, constant. When we proceed to the second approximation which, in de Pontécoulant's method, is really the discovery of particular integrals of the equations, it is theoretically unnecessary to introduce new arbitrary constants, the requisite number being already present in the complementary function. The relation $\mu = n^2a^3$ would, however, have been replaced by one much more complicated and certain important terms in the expressions for the coordinates would have been much less simple. It is possible to make the required changes in the arbitrary constants, *after* the expressions have been obtained, but we gain much by defining them as we proceed in the way finally necessary. To do so, the new constants must be retained and suitable meanings must be given to them.

In order to simplify the explanations as much as possible, we shall first neglect γ (or i) and therefore θ; we then merely treat the constants present in cases (i)—(iv) of Chap. VII. In cases (iii), (iv) no new arbitraries were introduced: we were only finding particular integrals. It is therefore only necessary to pay attention to those present in (i), (ii). As already stated in Art. 151, we might have taken the four cases at one step: the parts of $1/r$, v, due to the second approximation and collected in Art. 149, would then have been found together. Had we done so, the terms considered in (iii), (iv) would have appeared as parts of the solution which it is not necessary to consider here. We therefore treat cases (i), (ii) only and suppose that the second approximations, there separated, have been made together.

158. In these two cases there were five constants—α, b_0, δh, B, c_0—which were either new arbitraries introduced or parts of the assumed solutions which were not directly determinable. Between α, b_0 there was one relation, namely, equation (10) of Art. 134. But since μ was not present in the solution, another relation involving μ existed between α, b_0, δh. Hence only

three new *independent* arbitraries, which may be taken to be δh, B, c_0, were introduced. It is required to find what connection there is between these and the old arbitraries a, n, e, ϵ, ϖ.

The constant B arose exactly in the same way that ϵ did, namely, as the additive constant in the integration of the longitude equation. The constant additive to the longitude therefore becomes $B+\epsilon$ instead of ϵ. As both B, ϵ are arbitrary, the introduction of B is unnecessary and we can put it zero. The longitude is then expressible in the form

$$t \times \text{const.} + \epsilon + \text{periodic terms.}$$

Hence, in disturbed (or undisturbed) motion, ϵ *is the value of the mean* longitude at time* $t=0$.

The constant δh is determined in a similar way so that the meaning of n may be fixed, but a complication arises owing to the way in which δh occurs. The expression for dv/dt consists of a constant term and periodic terms. When the motion is undisturbed there is only one constant term and it was denoted by n. (The arbitrary actually introduced was h; n is the constant term in the development of h/r^2.) When the motion is disturbed, *definite* constant terms are present as well as the *arbitrary* constant δh. The expression for the mean longitude appears in the form

$$nt\,(1 + \text{powers of } m^2 \text{ etc.}) + \epsilon.$$

The presence of δh enables us to get rid of all these other terms, so that the longitude is expressible in the form

$$nt + \epsilon + \text{periodic terms.}$$

Hence n *is the mean angular velocity of the Moon* or, as it is generally called, *the mean motion*, whether the orbit be disturbed or undisturbed. Since the new definite terms which appear in the expression of dv/dt can always be eliminated in any stage of the approximations, n will be the mean motion at any stage and therefore in the final results.

It will be noticed that in case (i), δh was simply used to get rid of these new terms. In case (ii), the result obtained by equating to zero the new terms multiplied by t, was required in order to determine the new value of e. The introduction of δh in cases (iii), (iv) is unnecessary, for if we stop at the first powers of e', a/a', no definite constant terms arise in the new part of dv/dt; the value of δh would therefore have been found to be zero and it was neglected at the outset.

Next, to find b_0, we substituted in equation (8), Art. 131, assuming that the relation $\mu = n^2a^3$ still held. Had we simply used equation (1) without introducing α, this would have been unnecessary for the purpose of finding

* The mean value of any quantity expressed in this form is defined, in celestial mechanics, as its value when all the periodic terms are neglected.

b_0 which would have been determined by equation (10), Art. 134. But then the new relation between μ, a, n and the other constants would have to be determined from some equation of motion similar to (8) involving μ; a simple definition for a in disturbed motion is also required. It is found best to *define a by the equation $\mu = n^2a^3$, where n has the meaning just defined.* This we have done in case (i). The relation $\mu = n^2a^3$ having been assumed to hold and b_0 having been found from equation (8), the introduction of a is unnecessary *as far as the discovery of the solution is concerned*, but its presence is necessary in order to make it evident that the equation for δu, p. 96, is satisfied.

159. The constant ec_0 in case (ii), was the coefficient of $\cos\phi$ in the value assumed for $a\delta u$. Now the first approximation gave the value of a/r to be

$$1 + e\cos\phi,$$

neglecting powers of e higher than the first. Therefore, when we include in the value of a/r the results of the second approximation, the coefficient of $\cos\phi$ becomes

$$e(1 + c_0).$$

But e was an arbitrary of the first approximation to $1/r$ and ec_0 is an arbitrary introduced exactly in the same way in the second approximation to $1/r$. We can therefore determine c_0 at will. Instead of putting c_0 zero, it is found best to determine it so that *the coefficient of $\sin\phi$ in longitude is the same whether the motion be disturbed or undisturbed.* Certain definite terms will also occur with c_0 in this coefficient (p. 102); these can always be eliminated, by the proper use of c_0, at any stage of the approximations, in the same way that δh was used to cancel those occurring in the coefficient of t in the expression for the longitude.

Other methods for fixing the meaning of e have been used. The older lunar theorists, taking the true longitude as independent variable and expressing the time or the mean longitude and the other coordinates in terms of it, fixed the meaning of e so that the coefficient of the principal elliptic term, in the expression of the parallax in terms of the *true* longitude, was the same in disturbed and undisturbed motion. After the *true* longitude has been expressed in terms of the *mean* longitude (or of the time), the principal elliptic term in longitude contains powers of $m, e'^2, \ldots$ in its coefficient. The characteristic is, in any case, e. De Pontécoulant, when working out his theory, wished to compare his results with those of the earlier investigators. He therefore determined c_0 and e so that the coefficient of $\sin\phi$ in longitude was the same as with them. We have not followed him in this detail because the more complete theory of Delaunay has the definition of e used here and because e so defined is obtained observationally with much greater ease.

In working out with rectangular coordinates the inequalities considered in case (ii), I have defined the constant of eccentricity by the coefficient of $\sin\phi$ in the expression of the coordinate Y (Section iii, Chap. II.). This appeared to be the simplest plan, in view of the later approximations necessary to form a complete development by this method. See Section (ii), Chap. XI.

160. It is not necessary to introduce a new constant for the determination of ϖ in disturbed motion. The reason for this will be seen more clearly when we come to the method of the Variation of Arbitrary Constants as exemplified by Delaunay's theory. It will then be seen that the variable longitude of perigee, in disturbed motion, is expressed in the form

$$(1-c)\,nt + \varpi + \text{periodic terms.}$$

The constant ϖ is therefore the value of the mean longitude of perigee at time $t=0$ and it keeps this name in any theory; also $(1-c)\,n$ is the mean motion of the perigee. As the longitude of perigee only occurs in the elliptic expressions for the coordinates under the signs, sine and cosine, and as the periodic terms which occur in the above expression for the longitude of perigee have coefficients of the first order at least, we can, when substituting in the elliptic expressions for the coordinates, expand the sines or cosines so that no periodic terms appear in the arguments: the coordinates are therefore ultimately expressed as in Chap. VII. Since ϖ only occurs in the principal elliptic term of de Pontécoulant's method in the form $cnt+\epsilon-\varpi$, and since ϵ may be supposed to be known, ϖ may be defined by means of the value of this argument at time $t=0$.

It is usual to retain the term 'eccentricity' for e in disturbed motion whatever be the plan used to fix its meaning. The reason for giving to $(1-c)\,nt+\varpi$ the term 'mean longitude of perigee' will be evident from the remarks just made. The constants ϵ, ϖ are called the 'epoch of the mean longitude' and the 'epoch of the mean longitude of perigee,' respectively. The constant a, defined by the equation $\mu=n^2a^3$, is often called the 'mean distance' or the 'semi-major axis of the orbit.'

The term 'mean distance' thus applied to a is inexact according to the usual definition of the word 'mean' (see footnote, p. 118). With the above definition of a, we have determined r in the form

$$a/r = 1+\beta+\text{periodic terms,}$$

which would give $$r/a = 1+\beta'+\text{periodic terms} \qquad (a);$$

β, β' being constants depending on m, e'^2.... According to the definition of the word 'mean,' the mean distance should be $a(1+\beta')$, while $a/(1+\beta)$ is the distance corresponding to the mean value of the sine of the parallax. The latter is the quantity determined observationally and therefore of most importance in this connection. The terms in β are small and are easily found when the values of the other constants have been determined by observation.

161. Remarks quite similar to those made concerning e, ϖ, apply to γ, θ. We have determined γ in Art. 147 so that the coefficient of the principal term in z is the same as in undisturbed motion. It is better to define it so that the coefficient of the corresponding term in the *latitude* U, is the same as in undisturbed motion. The transformation from the old constant to the

new one is easily made when U has been found. The coefficient of $\sin\eta$ in U will be $\gamma(1+\beta'')$, where β'' depends on m, e^2, e'^2,.... We replace γ by $\gamma/(1+\beta'')$ wherever the former constant occurs; γ is called 'the tangent of the mean inclination.' The longitude of the node, when found as in Delaunay's theory, will be expressed in the form

$$(1-g)\,nt+\theta+\text{periodic terms.}$$

Hence $(1-g)\,n$ is the *mean motion of the Node*, θ is the *epoch of the mean longitude of the Node;* the latter is determined in de Pontécoulant's method by finding the value of the angle $gnt+\epsilon-\theta$ at time $t=0$.

Determination of the Constants by Observation.

162. The three coordinates of the Moon which are observed directly, are the longitude, the latitude and the parallax. Of these, an expression for the longitude has already been obtained; the expressions for the parallax and the latitude are deducible immediately.

Take the Earth's equatoreal radius as the unit of distance. Then $1/r$ will be the ratio of the Earth's equatoreal radius to the distance of the Moon, that is, the sine of the equatoreal horizontal parallax of the Moon. Let Π denote this parallax. We have approximately

$$\Pi=\sin\Pi+\tfrac{1}{6}\sin^3\Pi.$$

The average value of $\sin\Pi$ is about $\frac{1}{60}$. The error caused by neglecting the term $\frac{1}{6}\sin^3\Pi$ is about $\frac{1}{20{,}000}$ of the whole, corresponding to an error of $0''\cdot2$: this is within the limits of error of a single observation. To this degree of accuracy we can therefore put $\Pi=a/r$.

To find the latitude we have, since $\tan U=s$,

$$U=s-\tfrac{1}{3}s^3+\tfrac{1}{5}s^5-\ldots.$$

As s is a small quantity of the order γ, we can quickly find U when s is known.

These three angular coordinates are therefore expressible in the form

$$Q=A+Bt+\Sigma C\,{\cos \atop \sin}(\beta t+\beta') \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When Q denotes the longitude, the periodic terms are sines; when Q denotes the latitude, they are also sines and A, B are both zero; when Q denotes the parallax, we have $B=0$ and the periodic terms are cosines.

In all cases Q, and therefore A, C, are the circular measures of angles. To express them in degrees we multiply by $180/\pi$, or in seconds of arc by

$$180\times 60\times 60\div\pi=206{,}264\cdot8.$$

The number of seconds of arc in any coefficient is therefore obtained by inserting the numerical values of the constants and multiplying the result by 206,265.

163. Suppose that in the expressions for the coordinates, represented by the general form (1), we stop at a given order; they will then be reduced to a finite number of terms. If a number of values of Q, equal to the number of constants A, B, C, β, β' present, be given, each of these constants could be determined independently. But our expressions have shown that only six or, if we suppose μ unknown, only seven of these constants are independent. (We consider the solar constants known.) Hence, if the observations and the theory were both correct, exact relations ought to exist between the various constants thus found when the number of observed values of Q is greater than seven. But these conditions are not quite fulfilled. In the first place, each observation is only approximate and must be regarded as subject to error. In the second place, the coefficients of the periodic series, being each of them formally represented by an infinite series about the convergency of which we have no information, can only be considered at the best as approximate, apart from the question as to whether the infinite series is a correct representation of the coefficient. Assuming that the infinite series are possible and convergent, in order to determine the numerical values of the seven arbitraries, it is still necessary to choose the particular terms which are best adapted to our needs.

Now the methods used to find the constants present in an equation of the form (1) enable us, in general, to obtain with a high degree of accuracy the coefficient, period and argument of any term when the number of observed values of Q is very great.

Suppose that it be required to determine the constant e. We naturally choose out of one of the coordinates the term or terms in which a given alteration to e will produce the greatest effect on the value of that coordinate. This term is immediately recognised as being that with argument ϕ in the longitude. All the other terms in longitude containing e have either powers of e higher than the first in their characteristics, or e is multiplied by some small quantity such as m, γ^2; in parallax, all terms are multiplied by the small quantity $1/a$. Again, the number of available trustworthy observations of the longitude is far greater than those of the other coordinates. Finally, since we have chosen that the coefficient of $\sin\phi$ in longitude shall be the same as in elliptic motion, this coefficient can be obtained theoretically to any degree of accuracy we desire. For all these reasons the determination of e by observation from the term with argument ϕ in the longitude is the most suitable. The advantages of the definition of this constant, adopted in Art. 159, now become very evident.

It will easily be seen how the numerical values of all the seven arbitrary constants are determined. The values of n, ϵ are obtained from the non-periodic terms nt, ϵ in v. The principal elliptic term then furnishes e, ϖ and also cn if we wish to find the period by observation. The principal term in latitude—that with argument η $(=gnt+\epsilon-\theta)$—gives γ, θ and also gn. Finally, the constant part of $1/r$ furnishes the value of a; this constant part contains also a few terms depending on m, e'^2, etc. which are known, since their numerical values were previously found.

164. At the present day the numerical values of most of the constants are known with a very high degree of accuracy. Tables have thus been formed of the motion of the Moon from theory alone. Notwithstanding the great care bestowed by various investigators in including in them the results of all known causes, small differences between the tables and the observations are continually to be found. Some of these can be put down to errors of observation but many of them, especially when they exceed a certain limit and appear to be either periodic or secular, are due to imperfections either in the theory or in the numerical values of the constants used in the tables. Even when all the corrections due to known causes have been made, certain empirical corrections, not indicated by theory, have to be applied to these tables in order that they may agree with the observations. The tables published in 1857 by Hansen, together with certain corrections investigated by Newcomb (see Art. 173), are still used to obtain the places of the Moon given in the *Nautical Almanac* on pages iv to xii of each month.

A comparison of the values of cn, gn, as determined from theory and directly from observation, furnishes a valuable test of the sufficiency of the known causes to completely account for the motion of the Moon. The recorded observations of eclipses have enabled astronomers to obtain n, $(1-c)n$, $(1-g)n$ with a high degree of accuracy, and the values of c, g deduced therefrom agree very closely with those calculated by theory. Nevertheless, the small differences between theory and observation still leave something to be desired. As far as may be judged, the results deduced from observation appear to be rather more trustworthy than those deduced from theory.

In order to reduce the results of the preceding Chapter to numbers it is necessary to know beforehand the numerical values of certain of the constants. We take the units of time and length to be the sidereal day and the Earth's equatoreal radius, respectively: the numerical values of the constants used below are those by which Delaunay* reduced his theory to numbers (see Art. 173). He takes

$$2\pi/n' = 365{\cdot}25637 \text{ days}, \quad 1/a' = 8''{\cdot}75, \quad e' = 0{\cdot}01677106 \ldots\ldots (2).$$

The object of the following articles being merely to find the extent by which the principal inequalities affect the place of the Moon, we shall not here require to know the values of ϵ', ϖ' or of ϵ, ϖ, θ. The parts of chief importance are the coefficients and periods. In other words, we consider mainly the amplitudes and periods of the periodic oscillations and not their phases.

* *Mem. de l'Acad. des Sc.* Vol. XXIX. Chap. XI.

The Mean Period and the Mean Distance.

165. The longitude is expressed in the form

$$v = nt + \epsilon + \text{periodic terms.}$$

The mean longitude is therefore $nt + \epsilon$ and the Mean Period T is the time in which the mean longitude increases its value by 2π. Hence $T = 2\pi/n$. We find T directly from observation to be about $27\frac{1}{3}$ days or more exactly

$$2\pi/n = T = 27{\cdot}321661 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (3).$$

With the value of n' given in equations (2) of the previous Article, we deduce

$$m = n'/n = {\cdot}07480133 = 1/13\tfrac{1}{3}, \text{ approximately} \quad\ldots\ldots\ldots\quad (4).$$

The parallax is given (Art. 138) by

$$\frac{1}{r} = \frac{1}{a}\left(1 + \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4\right) + \text{periodic terms.}$$

The mean value of the sine of the equatoreal parallax is found directly from observation to be $3422''{\cdot}7$. To obtain $1/a$ we have therefore

$$\frac{1}{a}\left(1 + \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4\right) = 3422''{\cdot}7,$$

which, with the value of m just found, gives

$$1/a = 3419''{\cdot}6 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (5).$$

This value is very little altered by including the terms which have not been calculated here for the constant part of $1/r$.

The distance of the Moon corresponding to this value is 206,265/3419·6 equatoreal radii of the Earth or about 239,950 miles, taking the Earth's equatoreal radius as 3,978 miles. The real mean distance, calculated by the formula (α) of Art. 160, is 238,840 miles.

From the value (2) of a' and (5) of a, we deduce

$$\frac{a}{a'} = {\cdot}002559 = \tfrac{1}{391} \text{ approximately} \quad\ldots\ldots\ \ldots\ldots\ldots\ldots\quad (6).$$

The Variation.

166. The term with argument 2ξ in longitude or parallax is known by the name of the *Variation*. Let T be the mean periodic time. The Variation runs through all its values in time

$$2\pi/2\,(n - n') = T/2\,(1 - m) = 14\tfrac{3}{4} \text{ days, by (3), (4),}$$

or in half the mean synodic period of revolution.

The coefficient of this term in longitude was (Art. 138) seen to be

$$\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4,$$

which, when the value of m has been substituted and the result multiplied by 206,265, gives 34′ 51″. When the portions in the coefficient depending on higher powers of m and on e^2, e'^2 etc. are taken into consideration, the value of the coefficient is found by Delaunay to be 39′ 30″.

The investigations of case (i) have shown that if the approximations to the coefficients be continued, the result for the terms dependent on m only, will be

$$\frac{a}{r} = \Sigma b_{2q} \cos 2q\xi, \qquad v - n't - \epsilon' = \xi + \Sigma b'_{2q} \sin 2q\xi, \qquad (q = 0, 1, 2, \ldots),$$

where b_{2q}, b'_{2q} depend only on m and are each of the order m^{2q}. The terms considered in case (i) therefore constitute a curve, periodic with reference to axes moving in their own plane with uniform angular velocity n'. The curve, relatively to these axes, will be closed and symmetrical; the time of revolution round it will be $2\pi/(n - n')$, or a mean synodic month. Since, in the case of our Moon, all the coefficients b_{2q} are positive, the maximum and minimum values of a/r are given by $\xi = 0$ and $\xi = \pi/2$ respectively. Hence the shortest diameter of the oval is directed towards or away from the mean place of the Sun. We shall call this line the X-axis of the oval.

All the inequalities with arguments $2q\xi$ may be called 'Variational Inequalities' and the curve just defined the 'Variational Curve.' This curve has been calculated and drawn by Dr Hill for satellites of periods differing from that of the Moon.

He has shown* that as the value of m is increased, the ratio between the lengths of the two axes increases while the velocities at the ends of the longer axes, that is, in quadratures, diminish. For the value $m = 1/2{\cdot}78$ the velocities in quadratures vanish and the curve has cusps at those two points. M. Poincaré† has shown that if the value of m be still further increased the cusps are replaced by loops, so that a satellite whose mean period relative to that of the Sun is greater than $1/2{\cdot}78$, would appear in quadrature six times during one revolution.

The Parallactic Inequality.

167. The terms considered in case (iv) of the previous Chapter are closely related to the Variational inequalities by their arguments, the latter being simply odd instead of even multiples of ξ. The principal term—that of argument ξ—is called the *Parallactic Inequality.*

* *Amer. Jour. Math.* Vol. I. pp. 259, 260.

† *Méc. Cél.* Vol. I. p. 109.

The coefficient of this term in longitude was found in Art. 144 to be

$$-\left(\tfrac{15}{8}m + \tfrac{23}{8}m^2\right)\frac{a}{a'} = -1'\,48'',$$

by the values of m, a/a' given in (4) and (6) respectively. Delaunay's value of the whole coefficient is $2'\,7''$. According to the results of Chap. I. these numbers are to be multiplied by $(E-M)/(E+M) = 39/40$ approximately (Art. 168).

The period of the term is $2\pi/(n-n')$, that is, one mean synodic month.

Suppose that these inequalities be included with the Variational Inequalities in the expressions for a/r and v. We shall have

$$a/r = \Sigma b_q \cos q\xi, \quad v - n't - \epsilon' = \xi + \Sigma b'_q \sin q\xi;$$

where the terms for which q is even are functions of m only and those for which q is odd, besides being functions of m, have the factor

$$a(E-M)/a'(E+M).$$

When we put $-\xi$ for ξ, r is unaltered and $v - n't - \epsilon'$ changes sign. The curve, referred to the same moving axes as before, is therefore symmetrical about the line directed to the mean place of the Sun. In this case, however, $\xi = 0$ and $\xi = \pi$ do not give the same values for r. Let r_0, r_π denote the values of r when $\xi = 0$, π respectively. We then have

$$\frac{a}{r_0} - \frac{a}{r_\pi} = 2b_1 + 2b_3 + 2b_5 + \dots$$

$$= \text{a negative quantity}$$

in the case of our Moon, for b_1 is then greater than the sum of the quantities b_3, b_5 ... and it is negative (see equation (29), Art. 144). Hence

$$r_0 > r_\pi.$$

The longer X-axis is therefore directed towards the Sun and the shorter

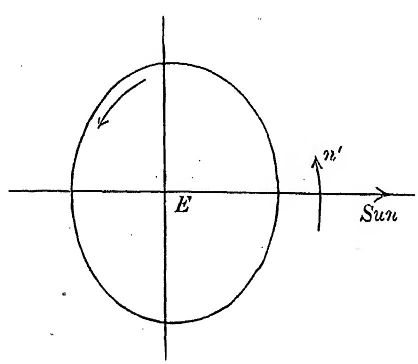

Fig. 7.

X-axis away from it. The effect of the inequalities depending on a/a' is to slightly distort the Variational curve in the direction of the Sun. The general character of the remarks just made, is not altered by the introduction of the squares and higher powers of a/a'.

All the inequalities which have arguments of the form $(2q+1)\xi$ may be called *Parallactic Inequalities.* The principal Parallactic Inequality has been used to determine the parallax of the Sun, that of the Moon being supposed known. It will be immediately seen that if we find the coefficient by observation and also by theory, we can, knowing m with great accuracy, deduce the value of $1/a'$. This method is, however, defective since it involves the accurate knowledge of the ratio $(E-M)/(E+M)$.

The variation as well as the other principal inequalities in the Moon's motion and the motions of the Apse and the Node, were first explained by Newton on the theory of gravity only. The values of their coefficients were obtained by him more or less approximately. The oval curve, called above the Variational Curve, was also recognised by him and the ratio of its two diameters was shown to be approximately as 69 : 70, corresponding to a coefficient 35′ 10″ of $\sin 2\xi$ in longitude. The results of Newton's investigations in the lunar theory are contained chiefly in Props. XXII. XXV.—XXXV. Book III. of the *Principia.*

168. The determination of the ratio M/E is a matter of some difficulty. There is no direct way of obtaining it. Probably the best method consists in finding the inequality in the motion of the Earth due to the action of the Moon. It will be readily seen from Chap. I. that E will describe a circle of radius $Ma/(E+M)$ about G, if we suppose that S describes a circle about G and that M describes a circle of radius a about E. As the Moon performs a synodic revolution, the apparent place of the Sun as seen from the Earth will therefore oscillate to and fro about its mean position. By observing the extent of this oscillation we can, knowing the other constants with considerable accuracy, deduce the value of $M/(E+M)$.

The constant may be determined by comparing the tides produced by the Moon with those produced by the Sun and also by comparing the observed nutation of the Earth's axis with the value deduced from theory. See de Pontécoulant, *Système du Monde*, Vol. IV., p. 651. The differences between the values so obtained, indicate that the numerical value of the ratio M/E is not certainly known within five per cent. of its true value.

The Principal Elliptic term, the Evection and the Mean Motion of the Perigee.

169. The Principal Elliptic term in longitude, having the same coefficient as in undisturbed motion, is, as far as quantities of the third order (Art. 50),

$$(2e - \tfrac{1}{4}e^3)\sin\phi.$$

The coefficient is found directly from observation to be

$$6^\circ\, 17'\, 19''\cdot 06 = 22639''\cdot 06.$$

Dividing this by 206,265 and equating the result to $2e - \frac{1}{4}e^3$, we obtain

$$e = \cdot 0548993.$$

The argument of the term is $cnt + \epsilon - \varpi$ and its period is therefore T/c. The value of c given by equation (25) of the last Chapter, is

$$c = 1 - \tfrac{3}{4}m^2 - \tfrac{225}{32}m^3.$$

Whence $$1 - c = \cdot 0041964 + \cdot 0029428 = \cdot 0071392.$$

Delaunay finds the complete value of $1 - c$ to be $\cdot 00845$. The period of the term is therefore about $27\frac{1}{2}$ days.

The term with argument $2\xi - \phi$ is called the *Evection*. The value of the coefficient in longitude was found (Art. 140) to be $+\frac{15}{4}me$ which, with the values of m, e deduced above, gives $52' 56''$. The complete value of this coefficient is $1° 16' 26''$.

The period of the term is

$$2\pi/(2n - 2n' - cn) = T/(1 - 2m + 1 - c)$$
$$= 31\tfrac{4}{5} \text{ days approximately.}$$

In Art. 160 we have seen that $(1 - c)n$ is the mean motion of the Perigee. The Perigee will therefore make a complete mean revolution in time $T/(1 - c) = 3827$ days with the value of c found above. The period as calculated by Delaunay* is $3232 \cdot 38$ days or about 9 years.

The class of inequalities defined by the arguments $2q\xi \pm p\phi$, (p, q integers) may be termed *Elliptic Inequalities*. The characteristic of the terms with arguments $2q\xi \pm p\phi$ is e^p.

170. We can combine the principal elliptic term and the Evection in a manner which enables us to illustrate the connection between the results obtained by the method of Chapter VII. and those obtained by causing the arbitrary constants to vary.

The principal elliptic terms in longitude and parallax are, neglecting quantities of the orders e^2 and m^2,

$$v = \ldots + 2e \sin \phi + \tfrac{15}{4}me \sin(2\xi - \phi) + \ldots,$$

$$\frac{a}{r} = \ldots + \quad e \cos \phi + \tfrac{15}{8}me \cos(2\xi - \phi) + \ldots.$$

Let $$2e_1 \sin \phi_1 = 2e \sin \phi + \tfrac{15}{4}me \sin(2\xi - \phi),$$

$$e_1 \cos \phi_1 = \quad e \cos \phi + \tfrac{15}{8}me \cos(2\xi - \phi).$$

* "Note sur les mouvements du périgée et du nœud de la Lune," *Comptes Rendus*, Vol. LXXIV. pp. 17—21.

From these we obtain, neglecting quantities of the order m^2,

$$e_1^2 = e^2 + \tfrac{15}{4} me^2 \cos(2\xi - 2\phi),$$

or

$$e_1 = e + \tfrac{15}{8} me \cos(2\xi - 2\phi).$$

We also deduce

$$e_1 \sin(\phi_1 - \phi) = \tfrac{15}{8} me \sin(2\xi - 2\phi).$$

Since e_1, e are of the same order, $\phi_1 - \phi$ must be a small quantity of the order m at least. To the order required we therefore have

$$\phi_1 - \phi = \tfrac{15}{8} m \sin(2\xi - 2\phi).$$

The transformations give

$$v = \ldots + 2e_1 \sin\phi_1 + \ldots, \qquad a/r = \ldots + e_1 \cos\phi_1,$$

where

$$e_1 = e + \tfrac{15}{8} me \cos(2\xi - 2\phi),$$

$$\phi_1 = \phi + \tfrac{15}{8} me \sin(2\xi - 2\phi) = nt + \epsilon - \{(1-c)\, nt + \varpi - \tfrac{15}{8} m \sin(2\xi - 2\phi)\}.$$

The effect of the action of the Sun as far as it produces the Evection, is to cause periodic variations of the eccentricity and of the longitude of perigee of the Moon. Had we solved by causing the arbitrary constants to vary, the variable values of the eccentricity and longitude of perigee would have been found to contain terms of this form.

In a similar manner, the other terms due to the action of the Sun may be included by assuming variable values for the constants. In order to perform the process completely, it would be necessary to assume that the velocities have the same form as in elliptic motion, in accordance with the principles of Chapter V. To the order considered in the above example this is easily seen to be true. We should of course include the terms depending on higher powers of e, e_1. This method of expressing the coordinates is not generally useful in itself: it is merely given here as an illustration. In fact, when we use the method of the variation of arbitrary constants, the reverse process has to be gone through to achieve the object in view, namely, the expression of each coordinate as a sum of periodic terms.

The Annual Equation.

171. The terms in longitude and parallax with argument $\phi' = n't + \epsilon' - \varpi'$ are known as the *Annual Equation*; their period is one year. The coefficient in longitude was found (Art. 142) to be

$$-e'(3m + 0m^2) = -12' \, 56'',$$

the complete value being $-11' \, 10''$.

The coefficient of the Annual Equation in parallax is, from the same Article,

$$-\frac{1}{a}e'\left(\tfrac{3}{2}m^2+0m^3\right)=0''\cdot 43,$$

which is the value correct to one hundredth of a second of arc.

The terms with arguments $2q\xi \pm p\phi'$ in longitude and parallax may be called the *Mean Period Inequalities.* The variable parts of their arguments depend only on the mean motions of the Sun and the Moon; they are independent of the longitudes of the perigee and the node of the Moon's orbit.

The Latitude and the Mean Motion of the Node.

172. We have found (Arts. 147, 162) as far as the order γm^2,

$$U=s=\left(1+\tfrac{1}{6}m^2\right)\gamma\sin\eta+\left(\tfrac{3}{8}m+\tfrac{25}{32}m^2\right)\gamma\sin(2\xi-\eta)+\tfrac{11}{8}m^2\gamma\sin(2\xi+\eta).$$

We shall replace the constant γ by that used in Delaunay's theory. In undisturbed motion let γ_1 be the sine of half the inclination; therefore, to the order considered here, $\gamma=2\gamma_1$. In disturbed motion, γ_1 is defined so that the coefficient of the principal term in U is the same as in undisturbed motion. Hence, in disturbed motion, we have to the order γm^2,

$$\gamma\left(1+\tfrac{1}{6}m^2\right)=2\gamma_1,\quad\text{or,}\quad \gamma=2\gamma_1\left(1-\tfrac{1}{6}m^2\right).$$

The value of U expressed in terms of γ_1 is then given by

$$U=2\gamma_1\sin\eta+\left(\tfrac{3}{4}m+\tfrac{25}{16}m^2\right)\gamma_1\sin(2\xi-\eta)+\tfrac{11}{8}m^2\gamma_1\sin(2\xi+\eta).$$

The coefficient of $\sin\eta$ is found from observation to be

$$5^\circ\, 7'\, 41''\cdot 26=18461''\cdot 26$$

corresponding to the value $\cdot 04475136$ of γ_1. The correct value is

$$\gamma_1=\cdot 04488663\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7),$$

the difference being due to the omission of the elliptic terms of higher orders in the coefficient of $\sin\eta$.

The period of the term is $2\pi/gn=T/g$. The value of g was found to be given (Art. 147) by

$$1-g=-\left(\tfrac{3}{4}m^2-\tfrac{9}{32}m^3-\tfrac{273}{128}m^4\right)=-\cdot 0040119,$$

the complete value being* $-\cdot 0040212$. The period is therefore about $27\tfrac{1}{5}$ days.

The mean motion of the node is $(1-g)n$. This being negative, the node moves backward, that is, in the direction opposite to that in which the Moon moves. It completes a revolution in time

$$T/(g-1)=6794\cdot 4 \text{ days or about } 18\tfrac{1}{2} \text{ years.}$$

* See footnote, p. 128.

It will be noticed that the first terms in the expressions for $1-c$, $g-1$ are the same. We have seen however that the apse moves forward twice as fast as the node moves backward. The difference is principally due to the near equality of the first two terms in the expression for $1-c$: the second term in the expression for $g-1$ is quite small.

The method of Art. 170 may be applied to the remaining terms in the expression for U, by assuming the inclination and the longitude of the node to be variable. The two equations necessary for their determination are furnished by the supposition that both U and dU/dt have the same form as in undisturbed motion.

173. The other terms present in the coordinates will not be examined here. Enough has been said to show how the effect of any particular term on the place of the Moon may be examined. The magnitudes of the coefficients can all be obtained in the manner explained above. These are best seen in a paper by Newcomb, *Transformation of Hansen's Lunar Theory**. A reference to it will show that there are 2 coefficients in longitude (those of the principal elliptic term and of the evection) which surpass 1°, 11 coefficients lying between 1° and 1′, 14 coefficients between 1′ and 10″; in latitude, 1 coefficient (that of the principal term) greater than 1°, 7 coefficients between 1° and 1′ and 6 between 1′ and 10″; in parallax, one term (the mean value) of nearly 1° in magnitude, one coefficient (that of the principal elliptic term) of just over 3′, and 7 coefficients lying between 35″ and 1″. The number of large coefficients is therefore not great, as far as the solar perturbations are concerned.

The methods used for deducing the numerical values of the constants from the recorded observations will be found in the various memoirs which contain determinations of these constants. The values of n', e' which have been employed in the preceding articles were obtained by Leverrier (*Ann. de l'Obs. de Paris, Mémoires*, Vol. IV.), those of e, γ by Airy (*Mem. of R. A. S.* Vol. XXIX.) and that of $1/a$ by Breen (*Mem. of R. A. S.* Vol. XXXII.). The values used by Hansen for the seven lunar constants will be found in the *Darlegung* (see Art. 202 below) and the *Tables de la Lune*. Later determinations have been made by Newcomb (*Papers published by the Commission on the Transit of Venus*, Pt III. and various memoirs in the first two volumes of the *Papers published for the use of the Amer. Eph.*). Further references will be found in the memoir mentioned in the previous paragraph, and in the *Nautical Almanac*, the *Connaissance des Temps*, the *American Ephemeris*, etc.

174. It is not difficult to verify the statement made in Art. 70, that c, g, found by Laplace's method with v as the independent variable, are the same as the values obtained when t is the independent variable.

In Laplace's method we modify the first approximation by substituting for ϖ, θ the values $(1-c)v+\varpi$, $(1-g)v+\theta$, respectively. If v' be the true longitude of the Sun, the disturbing function, which is expressed in terms of r, r', s, $v-v'$, will contain the angles

$$cv-\varpi, \quad n't+\epsilon'-\varpi', \quad gv-\theta, \quad v-v'.$$

In order to express $v-v'$, $n't+\epsilon'-\varpi'$ in terms of v, we have

$$v'=n't+\epsilon'+2e'\sin(n't+\epsilon'-\varpi')+\ldots$$

$$nt+\epsilon=v-2e\sin(cv-\varpi)+\ldots.$$

* *Astr. Papers for the use of the Amer. Eph.* Vol. I. pt. II. pp. 57—107.

Hence $v-v'$, $n't+\epsilon'-\varpi'$ can be expressed in terms of the four angles

$$v-\frac{n'}{n}v-\epsilon'+\frac{n'}{n}\epsilon, \quad \frac{n'}{n}v+\epsilon'-\varpi'-\frac{n'}{n}\epsilon, \quad cv-\varpi, \quad gv-\theta,$$

and therefore the disturbing function can be expressed in terms of v by means of the same four angles.

After solving the equations used in Laplace's method, we ultimately obtain

$$nt+\epsilon=v+K;$$

where K consists of a series of sines of the sums of multiples of the four angles, the coefficients being of the first order at least.

In order to get v in terms of t we must reverse this series. The first process in the reversion is to put $v=nt+\epsilon$ in the terms present in K. We then get

$$v=nt+\epsilon+K_1,$$

where K_1 consists of a series of sines whose arguments are made up of the four angles

$$(n-n')t+\epsilon-\epsilon', \quad n't+\epsilon'-\varpi', \quad cnt+c\epsilon-\varpi, \quad gnt+g\epsilon-g\varpi.$$

The next process is to substitute this new value of v in K and to expand the sines of the angles $cv-\varpi=cnt+c\epsilon-\varpi+cK_1$, etc. in powers of K_1. No new angles will be introduced. By continuing the process, v is ultimately expressed in terms of the four angles which, after putting for the arbitraries $c\epsilon-\varpi$, $g\epsilon-\theta$ the new arbitraries $\epsilon-\varpi$, $\epsilon-\theta$, are the same as those used in de Pontécoulant's theory. Also, v being expressed in the same form in both cases, c, g must have the same values.

CHAPTER IX.

THE THEORY OF DELAUNAY.

175. THIS Chapter will be devoted to an explanation of the manner in which Delaunay has applied the principles of the method of the Variation of Arbitrary Constants to the discovery of expressions for the coordinates which will represent the position of the Moon at any time. The principal object which Delaunay had in view and which he fully carried out, is stated in the preface to the two large volumes* containing his investigations, in the following words†:

'*Déterminer, sous forme analytique, toutes les inégalités du mouvement de la Lune autour de la Terre, jusqu'aux quantités du septième ordre inclusivement, en regardant ces deux corps comme de simple points matériels, et tenant compte uniquement de l'action perturbatrice du Soleil, dont le mouvement apparent autour de la Terre est supposé se faire suivant les lois du mouvement elliptique.*'

The limitations imposed on the problem are therefore the same as those made in the previous Chapters. The motion of the Sun is supposed to be elliptic and in the fixed plane of reference, the disturbing function is the same as that given in Art. 8, and the intermediate orbit is an ellipse obtained by neglecting the action of the Sun. No modification of the intermediate orbit, like that given in Chap. IV. and used in de Pontécoulant's and Laplace's theories, is necessary here.

The use of canonical systems of elements being the basis of Delaunay's theory, we shall depart from the notation used above and, after Art. 179, adopt that of Delaunay. The latter has the advantage of retaining a certain symmetry in the formulæ: it will also facilitate references to Delaunay's

* *Mém. de l'Acad. des Sc.* 4to Vols. XXVIII. (1860) 883 pp., XXIX. (1867) 931 pp. These will be referred to in this Chapter as 'Delaunay, I., II.'

† Delaunay, I. p. XXVI.

expressions and to the further developments (e.g. those in Chap. XIII.) which have been made according to his method by other investigators who have generally adopted his notation in their memoirs.

The method by which the transformations contained in Arts. 178, 185, 189, 190 below, are carried out, is not the same as that of Delaunay; the latter performed them by direct differentiation—a process somewhat tedious. Tisserand, in his account of the theory *, uses Jacobi's general dynamical methods, stated in Art. 94 above, to perform the transformations. The method used here is short and it has the advantage of showing immediately the terms which are to be added to R †.

176. In Chapters IV., V. have been given the principles on which the method of the variation of arbitrary constants is based. When the motion is undisturbed it is elliptic, and the coordinates are expressible in terms of the elements and of the time. When the action of the Sun is taken into account by considering the elements variable, it has been shown (Arts. 84–86, or 98) that the equations which express them in terms of the time are

$$\frac{d\beta_i}{dt} = \frac{\partial R}{\partial \alpha_i}, \quad \frac{d\alpha_i}{dt} = -\frac{\partial R}{\partial \beta_i}, \quad (i = 1, 2, 3) \quad \ldots\ldots\ldots\ldots\ldots(1).$$

In these, α_i, β_i have certain definite meanings with reference to the elliptic orbit: they are explained in Art. 92.

The equations in this form possess a serious defect. It will be remembered that R contains in its arguments, terms of the form $nt +$ const. Now (Art. 84) $n = \mu^{\frac{1}{2}} a^{-\frac{3}{2}} = \mu^{-1}(-2\beta_1)^{\frac{3}{2}}$. When, therefore, we form $\partial R/\partial \beta_1$, the time t will appear outside the signs sine and cosine and thus produce terms in the value of α_1, which increase continually with the time. It has been seen in Chap. IV. that such terms are to be eliminated if possible. The artifice used by Delaunay consists in simply replacing the variables β_1, α_1 by two others.

Before changing the variables to effect this object, some remarks must be made on the method of performing this and similar transformations required later.

177. *Method for transforming from one set of variables to another.*

Let *any* arbitrary variations $\delta\alpha_i$, $\delta\beta_i$ be given to α_i, β_i, and let δR be the corresponding change in R. We have then

$$\delta R = \Sigma \left(\frac{\partial R}{\partial \alpha_i} \delta\alpha_i + \frac{\partial R}{\partial \beta_i} \delta\beta_i \right), \qquad (i = 1, 2, 3).$$

* *Méc. Cél.* Vol. III. Chaps. XI., XII. Also, *Jour. de Liouville*, Vol. XIII. pp. 255—303.

† The method is used in a different way by Radau on pp. 336—340 of a paper "Remarques relatives à la Théorie des Orbites." *Bulletin Astronomique*, Vol. IX.

The six canonical equations (1) can therefore be written, as in Art. 98,

$$\Sigma\left(\frac{d\beta_i}{dt}\delta\alpha_i - \frac{d\alpha_i}{dt}\delta\beta_i\right) = \delta R,$$

or

$$\Sigma\,(d\beta_i\delta\alpha_i - d\alpha_i\delta\beta_i) = dt\delta R, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1'),$$

where R is supposed to be expressed in terms of α_i, β_i, t. The symbol d therefore denotes the *actual* change taking place in time dt, while δ denotes *any* arbitrary variation of the elements. If we wish to transform from α_i, β_i to another set of variables $\gamma_1, \gamma_2, \ldots \gamma_6$, the process of finding the new equations is rendered very easy. Suppose

$$\alpha_i = f_i(t, \gamma_1, \gamma_2, \ldots \gamma_6), \quad \beta_i = \mathrm{f}_i(t, \gamma_1, \gamma_2, \ldots \gamma_6);$$

we have, by the definitions of d, δ,

$$d\alpha_i = \frac{\partial f_i}{\partial t}dt + \frac{\partial f_i}{\partial\gamma_1}d\gamma_1 + \frac{\partial f_i}{\partial\gamma_2}d\gamma_2 + \ldots + \frac{\partial f_i}{\delta\gamma_6}d\gamma_6,$$

$$\delta\alpha_i = \frac{\partial f_i}{\partial\gamma_1}\delta\gamma_1 + \frac{\partial f_i}{\partial\gamma_2}\delta\gamma_2 + \ldots + \frac{\partial f_i}{\partial\gamma_6}\delta\gamma_6,$$

and similarly for the variables β_i.

The substitutions being made in the first member of (1′), we suppose R expressed in terms of the new variables, so that

$$\delta R = \frac{\partial R}{\partial\gamma_1}\delta\gamma_1 + \frac{\partial R}{\partial\gamma_2}\delta\gamma_2 + \ldots + \frac{\partial R}{\partial\gamma_6}\delta\gamma_6.$$

Equating the coefficients of the independent arbitraries $\delta\gamma_1, \delta\gamma_2, \ldots \delta\gamma_6$ to zero, we obtain the new set of equations satisfied by $\gamma_1, \gamma_2, \ldots \gamma_6$.

The transformations will in general be possible and definite if the Jacobian of α_i, β_i with respect to $\gamma_1, \gamma_2, \ldots \gamma_6$ does not vanish. For transformations in which the system remains canonical (Art. 87), see the works of Jacobi, Dziobek and Poincaré referred to in Art. 105. The formulæ for these are not of great value here because, in Delaunay's method, terms are added to R to keep the system canonical.

178. *Transformation to the variables* $w, \alpha_2, \alpha_3, L, \beta_2, \beta_3$.

Let $$\beta_1 = -\mu^2/2L^2, \qquad w = n(t + \alpha_1).$$

We have from Art. 84, $\beta_1 = -\mu/2a$ and $n^2 = \mu a^{-3}$. Hence

$$L = \sqrt{a\mu} = \mu(-2\beta_1)^{-\frac{1}{2}}, \qquad n = \mu^2/L^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The formulæ of transformation are

$$\beta_1 = -\mu^2/2L^2, \qquad \alpha_1 = wL^3/\mu^2 - t.$$

Whence $$\delta\beta_1 = \frac{\mu^2}{L^3}\delta L, \qquad \delta\alpha_1 = \frac{L^3}{\mu^2}\delta w + \frac{3wL^2}{\mu^2}\delta L,$$

$$d\beta_1 = \frac{\mu^2}{L^3}dL, \qquad d\alpha_1 = \frac{L^3}{\mu^2}dw + \frac{3wL^2}{\mu^2}dL - dt.$$

We have from these,

$$d\beta_1\delta\alpha_1 - d\alpha_1\delta\beta_1 = dL\delta w - dw\delta L + \frac{\mu^2}{L^3}dt\delta L;$$

and therefore from (1′), since the other variables remain unaltered,

$$dL\delta w + d\beta_2\delta\alpha_2 + d\beta_3\delta\alpha_3 - dw\delta L - d\alpha_2\delta\beta_2 - d\alpha_3\delta\beta_3 = dt\left(\delta R - \frac{\mu^2}{L^3}\delta L\right)$$
$$= dt\delta R_0,$$

where $$R_0 = R + \mu^2/2L^2 = R - \beta_1 \quad \text{.......................(3)}.$$

If we now suppose R_0 to be expressed in terms of the new variables, the equations remain canonical. They are

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial R_0}{\partial w}, & \frac{d\beta_2}{dt} &= \frac{\partial R_0}{\partial \alpha_2}, & \frac{d\beta_3}{dt} &= \frac{\partial R_0}{\partial \alpha_3}, \\ \frac{dw}{dt} &= -\frac{\partial R_0}{\partial L}, & \frac{d\alpha_2}{dt} &= -\frac{\partial R_0}{\partial \beta_2}, & \frac{d\alpha_3}{dt} &= -\frac{\partial R_0}{\partial \beta_3} \end{aligned}\right\}\text{..................(4)}.$$

It will be noticed that w is the mean anomaly in the undisturbed orbit (Chap. III.).

179. *Change of notation.*

We now take up the notation adopted by Delaunay and replace

$$w,\quad \alpha_2,\quad \alpha_3,\quad L,\quad \beta_2,\quad \beta_3,\quad R_0,\quad \gamma_1,\quad w',\quad \alpha_2',\quad \alpha_3'$$

by $$l,\quad g,\quad h,\quad L,\quad G,\quad H,\quad R,\quad \gamma,\quad l',\quad g',\quad h',$$

respectively.

The six canonical equations (4) may be written

$$dL\delta l + dG\delta g + dH\delta h - dl\delta L - dg\delta G - dh\delta H = dt\delta R \text{.........(4′)},$$

where

$$R = \frac{\mu^2}{2L^2} + \frac{n'^2a'^3}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - n'^2a'^3\frac{xx'+yy'+zz'}{r'^3}\text{...(5)}.$$

It will not be necessary to change the signification of n', a', e'—the constants referring to the solar orbit. Also n, a, e retain, for undisturbed motion, the meanings previously given.

According to the definitions given in Art. 92, h, g, l are, in undisturbed motion, the angular distances $x\Omega$, ΩA, AM_0 (fig. 4, p. 38); also H, G, L are double the rates of description of areas by the projection of the radius vector in the plane of reference, by the radius vector in the plane of the orbit, and by the uniformly revolving radius vector—supposed to be of length equal to the semi-major axis—in the plane of the orbit,

respectively. The correlation of areas to angles will be readily noticed. We have also, in undisturbed motion,

$$l+g+h = \text{mean longitude of Moon},$$
$$g+h = \text{longitude of perigee},$$
$$h = \text{longitude of node},$$
$$l = \text{mean anomaly},$$
$$\gamma = \text{sine of half the inclination}.$$

In the notation previously used, these were respectively denoted by

$$nt+\epsilon, \quad \varpi, \quad \theta, \quad w = nt+\epsilon-\varpi, \quad \sin\tfrac{1}{2}i.$$

It must be remembered that in de Pontécoulant's theory the letter γ was used to denote $\tan i$.

180. *The form of the development of R.*

The results of Art. 113 show that, after changing the notation, the arguments of all terms in R are expressible by means of the four angles

$$l, \quad l', \quad l+g, \quad h-l'-g'-h'.$$

The argument of any term is therefore of the form

$$il + i'g + i''h + i'''l' + q,$$

where i, i', i'', i''' may be any integers, positive, negative or zero, and

$$q = -i''(g'+h').$$

Since we suppose the perigee of the Sun's orbit fixed, q is a constant which vanishes when $i''=0$.

The coefficients of the development of R given in Art. 114 are functions of $e, \gamma, e', a/a'$ with the factor m^2/a; since $m = n'/n$, $n^2a^3 = \mu$, $n'^2a'^3 = m'$ *, they may be expressed as functions of a, e, γ, a', e'. Hence if we put $m'a^3/a'^3$, $l+g+h-l'-g'-h'$, l, l', $l+g$ for m^2, ξ, ϕ, ϕ', η, respectively and for γ^2 the expression $4\gamma^2(1-\gamma^2)(1-2\gamma^2)^{-2}$ we can, by expanding this latter quantity, deduce the form of the development required for this Chapter. This is, after adding the term $\mu^2/2L^2 = \mu/2a$,

$$R = \frac{\mu}{2a} + \frac{m'a^2}{a'^3}\Big[\tfrac{1}{4} + \tfrac{3}{4}\cos 2(l+g+h-l'-g'-h') - \tfrac{1}{2}e\cos l - \ldots + \tfrac{3}{4}e'\cos l'$$
$$+ \tfrac{3}{2}\gamma^2\cos 2(l+g) + \ldots + \tfrac{3}{8}\frac{a}{a'}\cos(h+g+l-h'-g'-l') + \ldots\Big] \ldots(6),$$

in which only the principal periodic terms have been written. Delaunay's development contains 320 periodic terms †.

* In Art. 114 we have put $\mu=1$ and m' for the ratio of the mass of the Sun to the sum of the masses of the Earth and the Moon.

† Delaunay, Vol. I. pp. 33—54.

If we neglect powers of γ^2 above the second, it is sufficient (remembering the change of notation) to put $4\gamma^2$ for γ^2 in the result of Art. 114. In the following, $l' = n't +$ const. and a', e', n', μ, m', g', h' are absolute constants.

From the relations given in Art. 84 we have, in the new notation,

$$\left.\begin{aligned} L &= \mu(-2\beta_1)^{-\frac{1}{2}} = na^2 = \sqrt{\mu a},\\ G &= h_0 = \sqrt{\mu a(1-e^2)},\\ H &= h_0 \cos i = (1-2\gamma^2)\sqrt{\mu a(1-e^2)}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (7),$$

and therefore

$$a = \frac{L^2}{\mu}, \quad e = \sqrt{\left(1 - \frac{G^2}{L^2}\right)}, \quad \gamma = \sqrt{\left(\tfrac{1}{2} - \tfrac{1}{2}\frac{H}{G}\right)} \quad \ldots\ldots\ldots (7');$$

by means of these relations the coefficients in R can be expressed in terms of L, G, H. We shall see later that it is not necessary to actually perform the substitutions.

Hence, as far as the elements of the Moon's orbit are concerned, *the arguments are functions of l, g, h only and the coefficients are expressible as functions of L, G, H only.*

It is to be noticed that R and the elliptic formulæ for v, $1/r$, U, given in the next Article, do not contain the time explicitly. It is only present in the initial development of R through its presence in l, l'.

181. *The elliptic expressions for the coordinates.*

The longitude v is immediately obtained from the expression given in Art. 50 if we replace therein, $nt + \epsilon$, w, η_0 by $l + g + h$, l, $l + g$, respectively, and to the order given, γ^2 by $4\gamma^2$. The value of $1/r$ is obtained from equation (16) of Art. 39 by expanding $J_i(ie)$ and replacing w by l. The latitude U can be deduced from the expression of Art. 50 by means of the equation,

$$U = s - \tfrac{1}{3}s^3 + \ldots.$$

In the expression for s, just referred to, we replace η_0 by $l + g$, w by l and γ by

$$2\gamma(1-\gamma^2)^{\frac{1}{2}}(1-2\gamma^2)^{-1} = 2\gamma(1 + \tfrac{3}{2}\gamma^2 + \ldots).$$

We thus obtain

$$\left.\begin{aligned} v &= h + g + l + (2e - \tfrac{1}{4}e^3)\sin l + \tfrac{5}{4}e^2 \sin 2l + \ldots\\ &\quad - \gamma^2 \sin(2g + 2l) - 2\gamma^2 e \sin(2g + 3l) + 2\gamma^2 e \sin(2g + l) + \ldots,\\ \frac{1}{r} &= \frac{1}{a}\left\{1 + (e - \tfrac{1}{8}e^3)\cos l + e^2 \cos 2l + \ldots \right\},\\ U &= 2\gamma(1-e^2)\sin(g + l) + 2\gamma e \sin(g + 2l) - 2\gamma e \sin g + \ldots\\ &\qquad - \tfrac{1}{3}\gamma^3 \sin 3(g + l) - \ldots \end{aligned}\right\} \ldots (8).$$

Delaunay gives the values of v, U correct to the sixth order and that of $1/r$ correct to the fifth order *.

* Delaunay, Vol. I. pp. 55—59. He uses V to denote the longitude.

Delaunay's method of Integration.

182. The method of integration adopted by Delaunay consists, in the first instance, in choosing out of R the constant term and one of the periodic terms, neglecting the rest of R. It is then found that the canonical equations can be integrated by means of series, and that the variables L, G, H, l, g, h can be expressed in terms of the time and of six new constants C, (G), (H), c, (g), (h). By means of these values, R is expressed in terms of the time and of the six new arbitraries. Having solved the equations by neglecting a portion of the disturbing function, enquiry is made in order to find what variable values these new arbitraries must have when this omitted portion of R is included. It is shown that by adding certain terms to the disturbing function, *the equations which express* C, (G), (H), c, (g), (h) *in terms of the time are canonical in form.* The power of the method arises from the fact that when the change in R from L, G, H, l, g, h to C, (G), (H), c, (g), (h) has been performed, *the periodic term considered has disappeared.*

The new arbitraries not being suitable for the purposes in view, two transformations will be made. Finally, we shall have formulæ similar to those from which we started, that is to say, the equations which express the new arbitraries in terms of the time are canonical, their relation to the new disturbing function is the same as before and the new disturbing function is of the same general form and has the same general properties as R.

Integration of the Canonical Equations when R is limited to one periodic term and the non-periodic portion.

183. The canonical equations, given by (4′) Art. 179, are

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial R}{\partial l}, & \frac{dG}{dt} &= \frac{\partial R}{\partial g}, & \frac{dH}{dt} &= \frac{\partial R}{\partial h}, \\ \frac{dl}{dt} &= -\frac{\partial R}{\partial L}, & \frac{dg}{dt} &= -\frac{\partial R}{\partial G}, & \frac{dh}{dt} &= -\frac{\partial R}{\partial H} \end{aligned}\right\} \quad \text{..............(4'')}.$$

Let $$R = -B - A\cos(il + i'g + i''h + i'''l' + q) + R_1,$$

or, putting $$il + i'g + i''h + i'''l' + q = \theta \quad \text{........................ (9)},$$

let $$R = -A\cos\theta - B + R_1,$$

in which $-B$ is the non-periodic part of R and $-A\cos\theta$ is any one of the periodic terms of R. *We shall first integrate the equations* (4″) *with* R_1 *neglected.*

Substituting in (4″) we obtain, since A, B are independent of l, g, h and θ is independent of L, G, H,

$$\left.\begin{aligned} \frac{dL}{dt} &= Ai\sin\theta, & \frac{dG}{dt} &= Ai'\sin\theta, & \frac{dH}{dt} &= Ai''\sin\theta, \\ \frac{dl}{dt} &= \frac{\partial A}{\partial L}\cos\theta + \frac{\partial B}{\partial L}, & \frac{dg}{dt} &= \frac{\partial A}{\partial G}\cos\theta + \frac{\partial B}{\partial G}, & \frac{\partial h}{dt} &= \frac{\partial A}{\partial H}\cos\theta + \frac{\partial B}{\partial H} \end{aligned}\right\} \text{(10)}.$$

The first three equations give

$$L = i\Theta, \qquad G = i'\Theta + (G), \qquad H = i''\Theta + (H) \ldots\ldots\ldots\ldots(11),$$

where (G), (H) are arbitrary constants and where

$$d\Theta/dt = A\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11').$$

Differentiating (9) with regard to t, we obtain

$$\frac{d\theta}{dt} = i\frac{dl}{dt} + i'\frac{dg}{dt} + i''\frac{dh}{dt} + i'''n',$$

which, by means of (10), becomes

$$\frac{d\theta}{dt} = \left(i\frac{\partial A}{\partial L} + i'\frac{\partial A}{\partial G} + i''\frac{\partial A}{\partial H}\right)\cos\theta + \left(i\frac{\partial B}{\partial L} + i'\frac{\partial B}{\partial G} + i''\frac{\partial B}{\partial H}\right) + i'''n'.$$

Now L, G, H and therefore A, B are, by (11), expressible as functions of one variable Θ. Hence, putting

$$B + i'''n'\Theta = B_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

there results, since $i = \partial L/\partial\Theta$, etc.,

$$\frac{d\theta}{dt} = \frac{dA}{d\Theta}\cos\theta + \frac{dB_1}{d\Theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

We have therefore the two simultaneous equations (11′), (13) for the expression of Θ, θ in terms of t. Multiply (13) by $d\Theta$ and (11′) by $d\theta$, subtract and integrate. We obtain

$$A\cos\theta + B_1 = C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

where C is an arbitrary constant. Whence

$$\sin\theta = \frac{1}{A}\sqrt{\{A^2 - (C - B_1)^2\}}.$$

Substituting in (11′) and integrating, we find, if c be an arbitrary constant,

$$t + c = \int^{\Theta} \frac{d\Theta}{\sqrt{\{A^2 - (C - B_1)^2\}}} \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The lower limit of the integral is taken to be the value of Θ when $\theta = 0$, that is, when $A = C - B_1$. Since A, B_1 are supposed to be known functions of L, G, H, and therefore, by (11), of Θ and of the constants (G), (H),

this integral gives the value of $t+c$ in terms of Θ and therefore of Θ in terms of $t+c$; hence L, G, H can be expressed in terms of $t+c$ and of the arbitraries C, (G), (H). The equation (14) then gives θ in terms of $t+c$.

184. The values of l, g, h are now to be found.

We have
$$\frac{dg}{dt}=\frac{\partial A}{\partial G}\cos\theta+\frac{\partial B}{\partial G}.$$

Substituting for dt from (15) and for $\cos\theta$ from (14), this becomes
$$\frac{dg}{d\Theta}=\left\{\frac{\partial A}{\partial G}\frac{C-B_1}{A}+\frac{\partial B}{\partial G}\right\}\div\sqrt{\{A^2-(C-B_1)^2\}}.$$

Also, from the way in which (G), (H) were introduced,
$$\frac{\partial A}{\partial G}=\frac{\partial A}{\partial (G)},\qquad \frac{\partial B}{\partial G}=\frac{\partial B_1}{\partial (G)}.$$

Hence, integrating,
$$g=\int\frac{\dfrac{\partial A}{\partial (G)}\dfrac{C-B_1}{A}+\dfrac{\partial B_1}{\partial (G)}}{\sqrt{\{A^2-(C-B_1)^2\}}}d\Theta+(g)$$

Similarly
$$h=\int\frac{\dfrac{\partial A}{\partial (H)}\dfrac{C-B_1}{A}+\dfrac{\partial B_1}{\partial (H)}}{\sqrt{\{A^2-(C-B_1)^2\}}}d\Theta+(h)\qquad\qquad(16).$$

Here (g), (h) are arbitrary constants; the upper limit of each integral is Θ and the lower limit is, as before, that value of Θ for which $A=C-B_1$.

Finally, as θ, g, h, l' are known in terms of t, we can find l from (9).

The three equations (15), (16) can be put into a more convenient form by assuming
$$K=\int\theta d\Theta=\int\cos^{-1}\frac{C-B_1}{A}d\Theta\qquad\qquad(17),$$

the upper and lower limits of the integral being the same as before. The three equations then become
$$t+c=-\frac{\partial K}{\partial C},\qquad g=\frac{\partial K}{\partial (G)}+(g),\qquad h=\frac{\partial K}{\partial (H)}+(h)\qquad\qquad(18);$$

for K may be considered to be a function of Θ, C, (G), (H).

The complete solution is contained in equations (9), (11), (14), (17), (18) which, after the elimination of θ, Θ, K, will give the values of L, G, H, l, g, h in terms of t and of the six new arbitraries C, (G), (H), c, (g), (h). The solution in this form is not, however, convenient for actual calculation; the method to be used will be outlined in Arts. 192, 193.

The Canonical Equations for C, (G), (H), c, (g), (h), when R_1, the portion of R previously omitted, is considered.

185. The canonical equations having been thus solved when R_1 is neglected, it is required to find the solution when R_1 is included. The solution just obtained contains six arbitraries C, (G), (H), c, (g), (h). We are going to inquire what *variable* values these six arbitraries must have when R_1 is not neglected. The method is then a further application of the principles of Chapter V., for we suppose L, G, H, l, g, h to have the same forms whether the new arbitraries be constant or variable. See Art. 98.

The canonical equations are given by

$$dL\delta l + dG\delta g + dH\delta h - dl\delta L - dg\delta G - dh\delta H$$
$$= dt\delta R = dt\delta(-A\cos\theta - B + R_1) \quad \text{.........(19).}$$

Into this equation we must substitute the values of L, G, H, l, g, h given by equations (11), (14), (18), the arbitraries C, (G), (H), c, (g), (h) being now considered variable. We suppose any arbitrary variation δ given to the latter and consider what changes are produced in R_1, L, etc.

From (12), (14) we deduce

$$-A\cos\theta - B = i'''n'\Theta - C.$$

The second member of (19) is therefore

$$i'''n'dt\delta\Theta - dt\delta C + dt\delta R_1.$$

For the first member we have, from (11),

$$\delta L = i\delta\Theta, \qquad \delta G = i'\delta\Theta + \delta(G), \qquad \delta H = i''\delta\Theta + \delta(H);$$
$$dL = id\Theta, \qquad dG = i'd\Theta + d(G), \qquad dH = i''d\Theta + d(H).$$

Substituting these and remembering that

$$\delta\theta = i\delta l + i'\delta g + i''\delta h,$$
$$d\theta = idl + i'dg + i''dh + i'''n'dt$$

the left-hand member of (19) becomes

$$d\Theta\delta\theta + d(G)\delta g + d(H)\delta h - d\theta\delta\Theta - dg\delta(G) - dh\delta H + i'''n'dt\delta\Theta.$$

But from the equations (18) we have

$$\delta g = \delta\frac{\partial K}{\partial(G)} + \delta(G), \qquad \delta h = \delta\frac{\partial K}{\partial(H)} + \delta(h),$$

with corresponding expressions for dg, dh. Substituting these values of δg, δh, dg, dh in the last expression for the left-hand member of (19), and

cancelling out the term $i'''n'dt\delta\Theta$ common to both members, we reduce equation (19) to

$$d\Theta\delta\theta - d\theta\delta\Theta + \left[d(G)\delta\frac{\partial K}{\partial(G)} + d(H)\delta\frac{\partial K}{\partial(H)} - \delta(G)d\frac{\partial K}{\partial(G)} - \delta(H)d\frac{\partial K}{\partial(H)}\right]$$
$$+ d(G)\delta(g) + d(H)\delta(h) - d(g)\delta(G) - d(h)\delta(H) = dt(\delta R_1 - \delta C) \ldots (20).$$

Since the operators d, δ are commutative, the part [...] may be written

$$\delta\left\{\frac{\partial K}{\partial(G)}d(G) + \frac{\partial K}{\partial(H)}d(H)\right\} - d\left\{\frac{\partial K}{\partial(G)}\delta(G) + \frac{\partial K}{\partial(H)}\delta(H)\right\}.$$

But by (17), K is a function of Θ, C, (G), (H) only, and each of these is now considered to be a function of t, owing to the variability of the six arbitraries. Hence

$$\delta K = \frac{\partial K}{\partial\Theta}\delta\Theta + \frac{\partial K}{\partial C}\delta C + \frac{\partial K}{\partial(G)}\delta(G) + \frac{\partial K}{\partial(H)}\delta(H),$$

and a corresponding expression of dK. The portion [...] therefore becomes

$$\delta\left\{dK - \frac{\partial K}{\partial\Theta}d\Theta - \frac{\partial K}{\partial C}dC\right\} - d\left\{\delta K - \frac{\partial K}{\partial\Theta}\delta\Theta - \frac{\partial K}{\partial C}\delta C\right\}.$$

This expression, by reason of equations (17), (18), is equal to

$$\delta\{-\theta d\Theta + (t+c)\,dC\} - d\{-\theta\delta\Theta + (t+c)\,\delta C\},$$

or, since $\delta t = 0$, $\theta\delta d\Theta = \theta d\delta\Theta$, etc., to

$$-d\Theta\delta\theta + dC\delta c + d\theta\delta\Theta - (dt + dc)\,\delta C.$$

If this be substituted in (20), the terms $d\Theta\delta\theta$, $d\theta\delta\Theta$, $dt\delta C$ disappear and the equation reduces to

$$dC\delta c + d(G)\delta(g) + d(H)\delta(h) - dc\delta C - d(g)\delta(G) - d(h)\delta(H) = dt\delta R_1 \ldots (21).$$

Supposing now that, by means of the values of L, G, H, l, g, h, the function R_1 has been expressed in terms of t and of C, (G), (H), c, (g), (h), we obtain from (21), in accordance with the remarks of Art. 177, the *canonical* equations

$$\left.\begin{aligned} \frac{dC}{dt} &= \frac{\partial R_1}{\partial c}, & \frac{d(G)}{dt} &= \frac{\partial R_1}{\partial(g)}, & \frac{d(H)}{dt} &= \frac{\partial R_1}{\partial(h)}, \\ \frac{dc}{dt} &= -\frac{\partial R_1}{\partial C}, & \frac{d(g)}{dt} &= -\frac{\partial R_1}{\partial(G)}, & \frac{d(h)}{dt} &= -\frac{\partial R_1}{\partial(H)} \end{aligned}\right\} \ldots\ldots (21').$$

A glance at the results obtained in Arts. 183, 184 will show that the variables in (21′) have meanings, with respect to the motion to which they refer, bearing a close analogy to those of α_i, β_i (Art. 176) with respect to elliptic motion. We shall see in Art. 187 that a transformation, similar to that of Art. 178, must be made.

The nature of the solution obtained in Arts. 183, 184.

186. Let us examine the nature of the solution obtained in Arts. 183, 184 when the variables are expressed in terms of the time and of the six arbitraries C, (G), (H), c, (g), (h): as R_1 is there neglected, these arbitraries are still considered to be constant. Since A is the coefficient of one of the periodic terms of the disturbing function, A, $\partial A/\partial\Theta$ are small quantities of the first order at least. Hence, the equations

$$\frac{d\Theta}{dt} = A \sin\theta, \qquad \frac{d\theta}{dt} = \frac{dA}{d\Theta}\cos\theta + \frac{dB_1}{d\Theta}$$

show that, if we neglect quantities of the first order and remember that A, B_1 are supposed to be expressed as functions of Θ and of the arbitraries (G), (H), a first approximation to the solution is given by

$$\Theta = \text{const.} = \Theta_0, \quad \theta = \theta_0(t + c),$$

in which Θ_0, c are arbitraries and θ_0 is a definite function of Θ_0, (G), (H).

A second approximation furnishes

$$\Theta = \Theta_0 + \Theta_1 \cos\theta_0(t + c), \qquad \theta = \theta_0(t + c) + \theta_1 \sin\theta_0(t + c).$$

Hence, assuming that developments in series are possible, the solution of these equations is given by

$$\left.\begin{aligned} \Theta &= \Theta_0 \qquad\quad\;\; + \Theta_1 \cos\theta_0(t + c) + \Theta_2 \cos 2\theta_0(t + c) + \dots \\ \theta &= \theta_0(t + c) + \;\theta_1 \sin\theta_0(t + c) + \;\theta_2 \sin 2\theta_0(t + c) + \dots \end{aligned}\right\} \quad \dots\dots (22).$$

Since $\theta = 0$ when $t = -c$, the arbitrary c is the same as that defined by (15). The arbitrary constant attached to Θ is Θ_0 which, by (15), may be expressed as a function of C, (G), (H); hence $\Theta_0, \Theta_1, \Theta_2, \dots, \theta_0, \theta_1, \theta_2 \dots$ are functions of C, (G), (H).

For the sake of brevity, denote by α_c, α_s the series

$$\alpha_1 \begin{matrix}\cos\\ \sin\end{matrix}\, \theta_0(t + c) + \alpha_2 \begin{matrix}\cos\\ \sin\end{matrix}\, 2\theta_0(t + c) + \dots$$

The solutions may then be written

$$\Theta = \Theta_0 + \Theta_c, \qquad \theta = \theta_0(t + c) + \theta_s \;\dots\dots\dots\dots\dots\dots (22').$$

With these values, the equations (18) give

$$g = (g) + g_0(t + c) + g_s, \qquad h = (h) + h_0(t + c) + h_s \;\dots\dots\dots (23),$$

where $g_0, g_1, \dots, h_0, h_1, \dots$ are functions of C, (G), (H). Therefore, from equation (9), we have

$$l = \frac{1}{i}\theta_0(t + c) - \frac{i'}{i}[(g) + g_0(t + c)] - \frac{i''}{i}[(h) + h_0(t + c)] - \frac{i'''}{i}l' - \frac{q}{i} + l_s \dots (23'),$$

where $l_1, l_2, \ldots$ are determined by

$$il_1 + i'g_1 + i''h_1 = \theta_1, \qquad il_2 + i'g_2 + i''h_2 = \theta_2, \qquad \ldots$$

Finally, the value of Θ in (22′), substituted in equations (11), gives

$$L = L_0 + L_c, \qquad G = G_0 + G_c, \qquad H = H_0 + H_c,$$

where

$$\left.\begin{array}{lll} L_0 = i\ \Theta_0, & L_1 = i\ \Theta_1, & L_2 = i\ \Theta_2, \quad \ldots \\ G_0 = i'\ \Theta_0 + (G), & G_1 = i'\ \Theta_1, & G_2 = i'\ \Theta_2, \quad \ldots \\ H_0 = i''\Theta_0 + (H), & H_1 = i''\Theta_1, & H_2 = i''\Theta_2, \quad \ldots \end{array}\right\} \ldots(24).$$

The coefficients $l_j, g_j, h_j, L_j, G_j, H_j$ being quantities of the order j at least, the above investigation shows that the new values of the elements can be formally expressed by series of cosines or sines of multiples of one angle $\theta_0(t+c)$, with coefficients in descending order of magnitude. The values of L, G, H, l, g, h may then be substituted in the disturbing function and in the elliptic expressions for the coordinates and the results expressed as sums of cosines or sines; the arguments may be freed from periodic terms in the manner explained in Art. 111.

187. *The form of the disturbing function after the substitutions have been made.*

Let us now consider the effect of the substitution of these values of L, G, H, l, g, h in R_1. Previously, R_1 consisted of periodic terms whose arguments were multiples of l, g, h, l', g', h' and whose coefficients were functions of L, G, H. After the substitutions have been made, R_1 will consist of periodic terms whose arguments, besides being multiples of l', g', h', contain multiples of

$$\theta_0(t+c), \qquad (g) + g_0(t+c), \qquad (h) + h_0(t+c);$$

θ_0, g_0, h_0, and the coefficients of these periodic terms, are functions of $C, (G), (H)$. Hence, when we commence to solve the equations (21′) by differentiating R_1 with respect to $C, (G), (H)$, the time $t+c$ will appear as a factor of the periodic terms. In order to avoid this, instead of $C, (G), (H), c, (g), (h)$, a new set of variables can be chosen which are such that the equations expressing them in terms of the time are still canonical; when R_1 has been expressed in terms of the new variables, it will have a form similar to that which R had, that is to say, R_1 will consist of periodic terms whose arguments contain multiples of three only of the variables and whose coefficients are functions of the three conjugate variables only. In making the transformation, the following Lemma will be required.

188. *Lemma.* *Let*

$$\phi = \theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25);$$

then $$\frac{1}{\theta_0} = \frac{\partial}{\partial C}(\Theta_0 + \tfrac{1}{2}\phi), \quad -\frac{g_0}{\theta_0} = \frac{\partial}{\partial (G)}(\Theta_0 + \tfrac{1}{2}\phi), \quad -\frac{h_0}{\theta_0} = \frac{\partial}{\partial (H)}(\Theta_0 + \tfrac{1}{2}\phi).$$

Differentiating (22′), we have

$$d\Theta = -\theta_0 dt\,[\Theta_1 \sin \theta_0 (t+c) + 2\Theta_2 \sin 2\theta_0 (t+c) + \ldots],$$

and $$\theta = \theta_0 (t+c) + \theta_1 \sin \theta_0 (t+c) + \theta_2 \sin 2\theta_0 (t+c) + \ldots;$$

whence $$\theta d\Theta = -\tfrac{1}{2}\theta_0 dt\,[\Theta_1\theta_1 + 2\Theta_2\theta_2 + 3\Theta_3\theta_3 + \ldots]$$
$$+ (t+c)\,dt \times \text{periodic terms} + dt \times \text{periodic terms.}$$

Therefore, since $K = \int \theta d\Theta$ and since K, $t+c$ are zero together, we obtain

$$K = -\tfrac{1}{2}\theta_0\phi\,(t+c) + (t+c) \times \text{periodic terms} + \text{periodic terms}\ldots(26).$$

Now K was a function of Θ, C, (G), (H) and Θ is a function of $(t+c)$, C, (G), (H). Denote by $\left(\frac{\partial K}{\partial C}\right)$, $\left(\frac{\partial K}{\partial (G)}\right)$, $\left(\frac{\partial K}{\partial (H)}\right)$ the partial differentials of K with respect to C, (G), (H), *after* the value of Θ has been substituted in K. We then have

$$\left(\frac{\partial K}{\partial C}\right) = \frac{\partial K}{\partial C} + \frac{\partial K}{\partial \Theta}\frac{\partial \Theta}{\partial C},$$

and similarly for (G), (H). Since $\partial K/\partial\Theta = \theta$, we obtain from these equations,

$$\frac{\partial K}{\partial C} = \left(\frac{\partial K}{\partial C}\right) - \theta\frac{\partial \Theta}{\partial C}, \quad \frac{\partial K}{\partial (G)} = \left(\frac{\partial K}{\partial (G)}\right) - \theta\frac{\partial \Theta}{\partial (G)}, \quad \frac{\partial K}{\partial (H)} = \left(\frac{\partial K}{\partial (H)}\right) - \theta\frac{\partial \Theta}{\partial (H)} \quad \ldots\ldots(27).$$

Now by (26),

$$\left(\frac{\partial K}{\partial C}\right) = -\tfrac{1}{2}(t+c)\frac{\partial}{\partial C}(\theta_0\phi)$$
$$+ \text{periodic terms having } (t+c)^0,\ (t+c)^1,\ (t+c)^2 \text{ as factors.}$$

Also, from (22′),

$$\theta\frac{\partial\Theta}{\partial C} = [\theta_0(t+c) + \theta_s]\left[\frac{\partial\Theta_0}{\partial C} + \frac{\partial\Theta_1}{\partial C}\cos\theta_0(t+c) + \ldots \right.$$
$$\left. -(t+c)\frac{\partial\theta_0}{\partial C}\{\Theta_1 \sin\theta_0(t+c) + 2\Theta_2 \sin 2\theta_0 (t+c) + \ldots\}\right]$$
$$= (t+c)\left[\theta_0\frac{\partial\Theta_0}{\partial C} - \frac{1}{2}\frac{\partial\theta_0}{\partial C}\phi\right]$$
$$+ \text{periodic terms having } (t+c)^0,\ (t+c)^1,\ (t+c)^2 \text{ as factors.}$$

Further, by (18), $$\partial K/\partial C = -(t+c).$$

Substituting these values for $\left(\frac{\partial K}{\partial C}\right)$, $\theta\frac{\partial\Theta}{\partial C}$, $\frac{\partial K}{\partial C}$ in the first of equations (27) and equating those coefficients of $-(t+c)$ which are independent of periodic terms, we find

$$1=\frac{1}{2}\frac{\partial}{\partial C}(\theta_0\phi)+\theta_0\frac{\partial\Theta_0}{\partial C}-\tfrac{1}{2}\phi\frac{\partial\theta_0}{\partial C}=\theta_0\frac{\partial}{\partial C}(\Theta_0+\tfrac{1}{2}\phi).$$

In a similar manner, by (18), (23),

$$\partial K/\partial(G)=g-(g)=g_0(t+c)+g_s,$$

and therefore, equating coefficients of $(t+c)$,

$$g_0=-\frac{1}{2}\frac{\partial}{\partial(G)}(\theta_0\phi)-\theta_0\frac{\partial\Theta_0}{\partial(G)}+\tfrac{1}{2}\phi\frac{\partial\theta_0}{\partial(G)}=-\theta_0\frac{\partial}{\partial(G)}(\Theta_0+\tfrac{1}{2}\phi),$$

with a corresponding equation for h_0.

189. *First change of variables in Equations* (21′) *to avoid the occurrence of terms increasing in proportion with the time.*

Put now

$$\left.\begin{array}{lll}\Lambda=\Theta_0+\tfrac{1}{2}\phi, & (G)=(G), & (H)=(H),\\ \lambda=\theta_0(t+c), & \kappa=(g)+g_0(t+c), & \eta=(h)+h_0(t+c)\end{array}\right\}\ldots(28),$$

and change the variables in (21′) or (21) from C, (G), (H), c, (g), (h) to Λ, (G), (H), λ, κ, η. The Lemma just proved gives

$$\frac{1}{\theta_0}=\frac{\partial\Lambda}{\partial C},\quad -\frac{g_0}{\theta_0}=\frac{\partial\Lambda}{\partial(G)},\quad -\frac{h_0}{\theta_0}=\frac{\partial\Lambda}{\partial(H)}.$$

We have also, from the relations (28),

$$\delta c=\frac{1}{\theta_0}\delta\lambda+\lambda\delta\frac{1}{\theta_0},\qquad dc=\frac{1}{\theta_0}d\lambda+\lambda d\frac{1}{\theta_0}-dt,$$

$$\delta(g)=\delta\kappa-\frac{g_0}{\theta_0}\delta\lambda-\lambda\delta\frac{g_0}{\theta_0},\quad d(g)=d\kappa-\frac{g_0}{\theta_0}d\lambda-\lambda d\frac{g_0}{\theta_0},$$

$$\delta(h)=\delta\eta-\frac{h_0}{\theta_0}\delta\lambda-\lambda\delta\frac{h_0}{\theta_0},\quad d(h)=d\eta-\frac{h_0}{\theta_0}d\lambda-\lambda d\frac{h_0}{\theta_0}.$$

Substituting, the first member of (21) becomes

$$\begin{aligned}&d(G)\delta\kappa+d(H)\delta\eta+dt\delta C-d\kappa\delta(G)-d\eta\delta(H)\\&+\left\{\frac{1}{\theta_0}dC-\frac{g_0}{\theta_0}d(G)-\frac{h_0}{\theta_0}d(H)\right\}\delta\lambda+\lambda\left\{dC\delta\frac{1}{\theta_0}-d(G)\delta\frac{g_0}{\theta_0}-d(H)\delta\frac{h_0}{\theta_0}\right\}\\&-\left\{\frac{1}{\theta_0}\delta C-\frac{g_0}{\theta_0}\delta(G)-\frac{h_0}{\theta_0}\delta(H)\right\}d\lambda-\lambda\left\{\delta Cd\frac{1}{\theta_0}-\delta(G)d\frac{g_0}{\theta_0}-\delta(H)d\frac{h_0}{\theta_0}\right\}.\end{aligned}$$

Substitute everywhere for $\frac{1}{\theta_0}$, $-\frac{g_0}{\theta_0}$, $-\frac{h_0}{\theta_0}$ their values $\frac{\partial \Lambda}{\partial C}$, $\frac{\partial \Lambda}{\partial (G)}$, $\frac{\partial \Lambda}{\partial (H)}$. The coefficient of $\delta\lambda$ becomes $d\Lambda$ and that of $d\lambda$ becomes $-\delta\Lambda$. The coefficient of $+\lambda$ is

$$dC\delta\frac{\partial\Lambda}{\partial C} + d(G)\,\delta\frac{\partial\Lambda}{\partial (G)} + d(H)\,\delta\frac{\partial\Lambda}{\partial (H)}$$

$$= \delta\left\{\frac{\partial\Lambda}{\partial C}\,dC + \frac{\partial\Lambda}{\partial (G)}\,d(G) + \frac{\partial\Lambda}{\partial (H)}\,d(H)\right\}$$
$$- \left\{\frac{\partial\Lambda}{\partial C}\,\delta dC + \frac{\partial\Lambda}{\partial (G)}\,\delta d(G) + \frac{\partial\Lambda}{\partial (H)}\,\delta d(H)\right\}$$

$$= \delta d\Lambda - \frac{\partial\Lambda}{\partial C}\,\delta dC - \frac{\partial\Lambda}{\partial (G)}\,\delta d(G) - \frac{\partial\Lambda}{\partial (H)}\,\delta d(H)$$

$=$ coefficient of $-\lambda$, since δ, d are commutative.

The equation (21) therefore reduces to

$$d(G)\,\delta\kappa + d(H)\,\delta\eta + dt\,\delta C - d\kappa\,\delta(G) - d\eta\,\delta(H) + d\Lambda\,\delta\lambda - d\lambda\,\delta\Lambda = dt\,\delta R_1 \qquad \text{......(29).}$$

Hence, putting $$R' = R_1 - C \qquad \text{.......................... (30),}$$

and supposing R' to be expressed in terms of Λ, (G), (H), λ, κ, η, we obtain

$$\left.\begin{aligned} \frac{d\Lambda}{dt} &= \frac{\partial R'}{\partial\lambda}, & \frac{d(G)}{dt} &= \frac{\partial R'}{\partial\kappa}, & \frac{d(H)}{dt} &= \frac{\partial R'}{\partial\eta}, \\ \frac{d\lambda}{dt} &= -\frac{\partial R'}{\partial\Lambda}, & \frac{d\kappa}{dt} &= -\frac{\partial R'}{\partial (G)}, & \frac{d\eta}{dt} &= -\frac{\partial R'}{\partial (H)} \end{aligned}\right\} \qquad \text{...........(29').}$$

The equations remain canonical and R' has a form similar to that which R had, namely, it is expressed by cosines of sums of multiples of λ, κ, η, l', g', h', with coefficients depending on Λ, (G), (H). The partials of R' with respect to the new variables will no longer introduce terms proportional to the time.

Second change of variables.

190. It is advisable to make a further transformation in order that when A (the coefficient of the periodic term previously considered) is put equal to zero, the new canonical equations shall reduce to the old ones (4″), Art. 183.

When A vanishes, it is easily seen from (22), (28), (23) that

$$\Theta = \Theta_0, \qquad \theta = \theta_0(t + c),$$

$$\Lambda = \Theta_0 = \frac{1}{i}L_0 = \frac{1}{i}L, \quad (G) = G - \frac{i'}{i}L, \qquad (H) = H - \frac{i''}{i}L,$$

$$\lambda = \theta_0(t + c) = \theta, \qquad \kappa = (g) + g_0(t + c) = g, \quad \eta = (h) + h_0(t + c) = h.$$

Put therefore

$$\left.\begin{aligned}\Lambda' &= i\Lambda, \qquad G' = i'\Lambda + (G), \qquad H' = i''\Lambda + (H),\\ \lambda' &= \frac{1}{i}\lambda - \frac{i'}{i}\kappa - \frac{i''}{i}\eta - \frac{i'''}{i}l' - \frac{1}{i}q, \qquad \kappa = \kappa, \qquad \eta = \eta\end{aligned}\right\} \text{.....(31)},$$

and transform from Λ, (G), (H), λ, κ, η to Λ', G', H', λ', κ, η.

We obtain by substitution in (29), after the elimination of C by means of (30),

$$\begin{aligned}dt\delta R' &= d\Lambda\delta\lambda + d(G)\delta\kappa + d(H)\delta\eta - d\lambda\delta\Lambda - d\kappa\delta(G) - d\eta\delta(H)\\ &= \frac{1}{i}d\Lambda'(i\delta\lambda' + i'\delta\kappa + i''\delta\eta) + \left(dG' - \frac{i'}{i}d\Lambda'\right)\delta\kappa + \left(dH' - \frac{i''}{i}d\Lambda'\right)\delta\eta\\ &\quad - (id\lambda' + i'd\kappa + i''d\eta + i'''n'dt)\frac{1}{i}\delta\Lambda' - d\kappa\left(\delta G' - \frac{i'}{i}\delta\Lambda'\right) - d\eta\left(\delta H' - \frac{i''}{i}\delta\Lambda'\right)\\ &= d\Lambda'\delta\lambda' + dG'\delta\kappa + dH'\delta\eta - d\lambda'\delta\Lambda' - d\kappa\delta G' - d\eta\delta H' - \frac{i'''}{i}n'dt\delta\Lambda'.\end{aligned}$$

Hence, by putting $$R'' = R' + \frac{i'''}{i}n'\Lambda' \text{ .... (31')},$$

the equations can be written in the canonical form

$$\left.\begin{aligned}\frac{d\Lambda'}{dt} &= \frac{\partial R''}{\partial\lambda'}, \qquad \frac{dG'}{dt} = \frac{\partial R''}{\partial\kappa}, \qquad \frac{dH'}{dt} = \frac{\partial R''}{\partial\eta},\\ \frac{d\lambda'}{dt} &= -\frac{\partial R''}{\partial\Lambda'}, \qquad \frac{d\kappa}{dt} = -\frac{\partial R''}{\partial G'}, \qquad \frac{d\eta}{dt} = -\frac{\partial R''}{\partial H'}\end{aligned}\right\} \text{.........(32)},$$

and these new variables will reduce to the old ones when A is put zero. It is evident that the remarks made at the end of the previous article will also apply to these equations.

191. We will now see how the new variables are related to those found in the solution given in Art. 186; also, how the new disturbing function R'' is related to R.

Let $$\phi' = \theta_1 L_1 + \theta_2 L_2 + \ldots = i\phi,$$

by (11), (25). The equations (31), (28), (24) furnish

$$\Lambda' = L_0 + \tfrac{1}{2}\phi', \qquad G' = G_0 + \tfrac{1}{2}\frac{i'}{i}\phi', \qquad H' = H_0 + \tfrac{1}{2}\frac{i''}{i}\phi',$$

$$\lambda' = (l) + l_0(t + c), \qquad \kappa = (g) + g_0(t + c), \qquad \eta = (h) + h_0(t + c),$$

where, by (23'),

$$(l) + l_0(t + c) \equiv \frac{1}{i}\theta_0(t + c) - \frac{i'}{i}\kappa - \frac{i''}{i}\eta - \frac{i'''}{i}l' - \frac{1}{i}q.$$

The new variables λ', κ, η are therefore nothing else than those non-periodic parts of l, g, h which were obtained by the solution of the equations (4'').

Again, we have

$$\begin{aligned} R &= -A\cos\theta - B + R_1 \\ &= -A\cos\theta - B + C + R', && \text{by (30)}, \\ &= -A\cos\theta - B + C - \frac{i'''}{i}n'\Lambda' + R'', && \text{by (31')}. \end{aligned}$$

But since $\qquad \Lambda' = i\Lambda = i(\Theta_0 + \tfrac{1}{2}\phi),$

we obtain $\qquad R'' = R_1 - C + i'''n'(\Theta_0 + \tfrac{1}{2}\phi);$

that is, R'' is the same as R_1 except for some additional non-periodic terms. *The periodic term $A\cos\theta$ is therefore not present in the new disturbing function.* Since R_1 does not contain $-A\cos\theta$, the last equation also shows that, when we make the change of variables in R by giving to L, G, H, l, g, h their values as functions of $\Lambda', G', H', \lambda', \kappa, \eta$, the periodic term $A\cos\theta$ in R will identically vanish (See Art. 197). This furnishes a means of verifying the calculations.

The application of the previous results to the calculation of an operation.

192. It is necessary to see how the results obtained in the previous articles are applied in the actual calculations. In the first place, R was expressed in terms of a, e, γ, l, g, h. Out of R the terms $-B-A\cos\theta$ were chosen: of these, $-B$ is the whole of the non-periodic part of R and $-A\cos\theta$ is any one periodic term. In the canonical equations (4″) for L, G, H, l, g, h, we insert this value of R.

Equations (7) give the values of L, G, H in terms of a, e, γ. By means of them, we find from the integrals

$$G = \frac{i'}{i}L + (G), \qquad H = \frac{i''}{i}L + (H),$$

any two of the quantities a, e, γ in terms of the third, say a, γ in terms of e, and substitute in the expression in (7) for L, which then becomes a function of $e, (G), (H)$ only. From this, dL/dt is deduced as a function of $de/dt, e, (G), (H)$.

But $dL/dt = \partial R/\partial l$, where $R = -A\cos\theta - B$. Whence, after expressing A in terms of $e, (G), (H)$, we find

$$de/dt = \sin\theta \times \text{function of } e, (G), (H) \ldots\ldots\ldots\ldots\ldots (33).$$

Also

$$\frac{d\theta}{dt} = i\frac{dl}{dt} + i'\frac{dg}{dt} + i''\frac{dh}{dt} + i'''n' = -i\frac{\partial R}{\partial L} - i'\frac{\partial R}{\partial G} - i''\frac{\partial R}{\partial H} + i'''n'.$$

Now $\frac{\partial R}{\partial L}$, $\frac{\partial R}{\partial G}$, $\frac{\partial R}{\partial H}$ may be expressed in terms of $\frac{\partial R}{\partial a}$, $\frac{\partial R}{\partial e}$, $\frac{\partial R}{\partial \gamma}$, a, e, γ, by means of the relations (7), (7′). We therefore obtain

$$\frac{d\theta}{dt} = \text{function of } a, e, \gamma + \cos\theta \times \text{function of } a, e, \gamma,$$

or, according to the remarks made above,

$$\frac{d\theta}{dt} = \text{function of } e, (G), (H) + \cos\theta \times \text{function of } e, (G), H \;\dots (34).$$

The equations (33), (34) involve only the two dependent variables e, θ and the independent variable t. They are equivalent to (11′), (13) and are integrated as in Art. 186 by continued approximation or by some similar method, giving

$$\left.\begin{aligned} e^2 &= e_0^2 \qquad\quad + \text{cosines of multiples of } \theta_0(t+c), \\ \theta &= \theta_0(t+c) + \text{sines of multiples of } \theta_0(t+c) \end{aligned}\right\}\dots\dots\dots\dots(35),$$

where e_0, c are the two arbitrary constants and θ_0 is a function of e_0, (G), (H).

In certain operations where e appears as a denominator in (34), it is found to be more convenient to solve (33), (34) by finding

$$\left.\begin{aligned} e\cos\theta &= \text{const.} + \text{cosines of multiples of } \theta_0(t+c), \\ e\sin\theta &= \qquad\qquad \text{sines of multiples of } \theta_0(t+c) \end{aligned}\right\}\dots\dots(35').$$

Here the arbitrary e_0 is the coefficient of $\sin\theta_0(t+c)$ and it does not appear in the right-hand members as a denominator. See Art. 196.

From these we can deduce the values of a, γ^2 (which were found in terms of e, (G), (H)), expressed as functions of e_0, (G), (H) and cosines of multiples of $\theta_0(t+c)$. In all cases the coefficients of the sines and cosines are functions of e_0, (G), (H) only. Let a_0, γ_0^2 be the non-periodic terms of a, γ^2 and eliminate (G), (H). We have then

$$\left.\begin{aligned} a &= a_0 + \text{functions of } a_0, e_0, \gamma_0^2 \text{ multiplied by} \\ &\qquad\qquad\qquad \text{cosines of multiples of } \theta_0(t+c), \\ e^2 &= e_0^2 + \text{similar terms}, \\ \gamma^2 &= \gamma_0^2 + \text{similar terms} \end{aligned}\right\}\dots(36).$$

When the form (35′) is used, the expression for e^2 contains some other non-periodic terms.

193. Next, equate $-\frac{dl}{dt}$, $-\frac{dg}{dt}$, $-\frac{dh}{dt}$ to the values of $\frac{\partial R}{\partial L}$, $\frac{\partial R}{\partial G}$, $\frac{\partial R}{\partial H}$,

already found. Eliminating a, e, γ, θ from $\partial R/\partial L$, etc. by means of (35) or (35′), (36) and integrating, we obtain

$$\left.\begin{aligned} l &= \lambda' + \text{functions of } a_0, e_0, \gamma_0 \text{ multiplied by} \\ &\qquad\qquad \text{sines of multiples of } \theta_0 (t+c), \\ g &= \kappa + \text{similar terms} \\ h &= \eta + \text{similar terms} \end{aligned}\right\} \ldots (37).$$

Here λ', κ, η are the non-periodic parts of l, g, h; also, by means of the relation $\theta = il + i'g + i''h + i'''l' + q$, we can express $\theta_0 (t+c)$, which is simply the non-periodic part of θ, in terms of λ', κ, η, l', q.

The results (36), (37) contain the required solution of the equations.

In order to prepare for the next operation, we substitute these results in R (and also in the expressions for the longitude, latitude and parallax, previously obtained); R will then become a function of a_0, e_0, γ_0, λ', κ, η.

Our new variables will be Λ', G', H', λ', κ, η,

where $$\Lambda', G', H' = \text{functions of } a_0, e_0, \gamma_0 \ldots\ldots\ldots\ldots\ldots\ldots (38),$$

found by means of the relations given in Art. 191. For since a, e, γ are given by (36), the values of L, G, H can be deduced; each of them will be expressed by a constant term and by cosines of multiples of $\theta_0 (t+c)$, the former and the coefficients of the latter being functions of a_0, e_0, γ_0. We then have L_0, G_0, H_0; the function ϕ' is deduced from the series obtained for θ, L.

Finally, since (Art. 191)

$$\begin{aligned} R'' &= R + A\cos\theta + B - C + \frac{i'''}{i} n'\Lambda' \\ &= R - i''' n'\Theta + \frac{i'''}{i} n'\Lambda', &&\text{by (12), (14),} \\ &= R - \frac{i'''}{i} n'(L - \Lambda'), &&\text{by (11),} \end{aligned}$$

we can obtain the new disturbing function. It will be found that, when R and L have been expressed in terms of the new variables, the term $-A\cos\theta$ will not be present in R''.

Taking the new variables and the new disturbing function the equations for them are still canonical. Also, as (38) gives the values of Λ', G', H' in terms of a_0, e_0, γ_0, we are in a position to go through the whole process again with another periodic term. Since the letters L, G, H, a, e, γ, l, g, h, R have disappeared completely, there is no further need of the symbols Λ', G', etc.: as soon as the operation is finished, they are replaced for simplicity by the letters L, G, etc. During the process, it is frequently more convenient to use n instead of a: the relation between n, a is always defined by the equation $\mu = n^2 a^3$.

Particular cases.

194. It is evident that if any or all of the integers i', i'', i''' are zero, the same methods will hold; the only difference will be a greater simplicity in the results. When i' or i'' is zero, we get

$$G=(G)=G_0 \text{ or } H=(H)=H_0,$$

respectively. When i''' is zero, the new part to be added to R vanishes.

When i is zero but either i' or i'' not zero, the method requires a slight modification. Suppose i' unequal to zero. We assume $G=i'\Theta$ instead of $L=i\Theta$; the solution evidently proceeds on exactly the same lines since, in the first instance, the equations were symmetrical with respect to L, G, H.

195. When i, i', i'' are all zero, we can modify the solution in a way which saves several operations.

The angle θ is reduced to $i'''l'$, for $q=0$ when $i''=0$. We take out of R, instead of a single term, the terms

$$-A_1\cos l' - A_2\cos 2l' - A_3\cos 3l' - \ldots = -\Sigma A_p\cos pl'.$$

When this is substituted in the canonical equations (4″) we obtain

$$\frac{dL}{dt}=0, \qquad \frac{dG}{dt}=0, \qquad \frac{dH}{dt}=0,$$

$$\frac{dl}{dt}=\Sigma\frac{\partial A_p}{\partial L}\cos pl', \qquad \frac{dg}{dt}=\Sigma\frac{\partial A_p}{\partial G}\cos pl', \qquad \frac{dh}{dt}=\Sigma\frac{\partial A_p}{\partial H}\cos pl'.$$

Hence L, G, H are constant and therefore

$$l=(l)+\Sigma\frac{1}{pn'}\frac{\partial A_p}{\partial L}\sin pl' \quad\text{.......................(39)},$$

with similar expressions for g, h.

It is necessary to see what the new canonical equations become when R_1 is not neglected. They are expressed by the equation

$$dt\delta(R_1-\Sigma A_p\cos pl')=dL\delta l+dG\delta g+dH\delta h-dl\delta L-dg\delta G-dh\delta H\ldots(40).$$

Since L, G, H remain unaltered, take L, G, H, (l), (g), (h) as new variables. Substituting from (39) for l, g, h, the second member of (40) becomes

$$dL\delta(l)+dG\delta(g)+dH\delta(h)-d(l)\delta L-d(g)\delta G-d(h)\delta H$$
$$+\Sigma\frac{1}{pn'}\sin pl'\left[dL\delta\frac{\partial A_p}{\partial L}+\ldots+\ldots-\delta Ld\frac{\partial A_p}{\partial L}-\ldots-\ldots\right]$$
$$-\Sigma\cos pl'\left[\frac{\partial A_p}{\partial L}\delta L+\frac{\partial A_p}{\partial G}\delta G+\frac{\partial A_p}{\partial H}\delta H\right]dt.$$

The second line of this expression can be shown to vanish in the same way as a similar expression considered in Art. 189, p. 148; the third line is equal to

$$-\Sigma\delta A_p \cos pl' dt = -dt\delta[\Sigma A_p \cos pl'],$$

and it therefore disappears with the same term present in the first member of (40). Hence the first line is equal to $dt\delta R_1$, and when R_1 has been expressed in terms of L, G, H, (l), (g), (h), the equations for these new variables will be canonical.

The rule for transformation is simple. We solve and find

$$l=(l)+\Sigma l_p \sin pl', \qquad g=(g)+\Sigma g_p \sin pl', \qquad h=(h)+\Sigma h_p \sin pl',$$

where l_p, g_p, h_p are functions of a, e, γ, or of L, G, H. All that is necessary is to replace l by $l+\Sigma l_p \sin pl'$, with similar changes for g, h, in R, v, U, $1/r$. The new equations for L, G, H, l, g, h are canonical, they reduce to their former values when all the coefficients A_p are zero and, after the change of variables has been made, the terms $-\Sigma A_p \cos pl'$ will not be present in the new disturbing function.

It will be noticed that the constant term is not included here. Since it is a function of L, G, H (which were shown to be constants) only, it can produce no new parts depending on l, g, h and it need not, in this case, be considered.

196. It has been mentioned in Art. 192 that, when the equations (33), (34) have been prepared for integration, they sometimes take the forms

$$\left.\begin{aligned} \frac{de}{dt} &= M(1+\Sigma M_i e^{2i}) \sin\theta, \\ \frac{d\theta}{dt} &= N(1+\Sigma N_i e^{2i}) + \frac{M}{e}(1+\Sigma P_i e^{2i}) \cos\theta, \end{aligned}\right\} (i=1, 2, \ldots);$$

here M is of the second order at least and N, M_i, N_i, P_i are of order zero; these coefficients are supposed to be independent of e, t, θ, so that, as far as the integration of the equations is concerned, they are constants. If we integrate in series by continued approximation, difficulties may arise owing to the presence of e as the denominator of a fraction. This may be avoided as follows.

We deduce from the equations just given,

$$\frac{d}{dt}(e\cos\theta) = M(\Sigma M_i e^{2i-2} - \Sigma P_i e^{2i-2})(e\sin\theta)(e\cos\theta) - N(1+\Sigma N_i e^{2i})(e\sin\theta),$$

$$\frac{d}{dt}(e\sin\theta) = M + M[(e\sin\theta)^2 \Sigma M_i e^{2i-2} + (e\cos\theta)^2 \Sigma P_i e^{2i-2}] + N(1+\Sigma N_i e^{2i})(e\cos\theta).$$

Since the least value of i is unity, e does not occur as a denominator in these expressions; also, the second members are expressible by integral powers of $e\cos\theta$, $e\sin\theta$.

To solve these equations assume

$$\begin{aligned} e\cos\theta &= E_0 + E_1 \cos\theta_0(t+c) + E_2 \cos 2\theta_0(t+c) + \ldots, \\ e\sin\theta &= \phantom{E_0 +{}} E_1' \sin\theta_0(t+c) + E_2' \sin 2\theta_0(t+c) + \ldots. \end{aligned}$$

If we put $E_1' = e_0$, it can be shown that E_i, E_i' are of the order $e_0{}^i$ at least; also, $E_2 = E_2'$, $E_3 = E_3'$, etc. The arbitrary constants are e_0, c: the quantity θ_0 is a certain definite function of M, N, M_i, N_i, P_i, $e_0{}^2$ and it does not contain e_0 as a denominator.

In the cases where e appears as a denominator, it is found, when we proceed to the substitution of the values of θ, e in the expressions for R and for the coordinates, that we only require to know $e \cos \theta$, $e \sin \theta$, e^2 and powers and products of these quantities; since their values do not contain e_0 as a denominator, no difficulty will ensue. See Delaunay, I. pp. 107, 108, 878—882 and Tisserand, *Méc. Cel.* Vol. III. pp. 216—220.

The general plan of procedure.

197. A general view of the whole process will perhaps make the comprehension of Delaunay's method easier.

We find first the elliptic values of the coordinates of the Sun and the Moon and, by means of them, express R in terms of l, g, h, a, e, γ, referring to the Moon, and of l', $g' + h'$, a', e', referring to the Sun. We have also definite relations between L, G, H and a, e, γ.

Choosing out of R the non-periodic part and *one* of the periodic terms, we solve the canonical equations for these portions of R and find the values of l, g, h, a, e, γ in terms of the time and of λ', κ, η, a_0, e_0, γ_0; of the latter, λ', κ, η contain three new arbitraries which are the constant parts of l, g, h: the other three new arbitraries are a_0, e_0, γ_0. These values are substituted in R, v, $1/r$, U which then become functions of λ', κ, η, a_0, e_0, γ_0. Certain terms are added to R and, when the change of variables has been made, it is found that the periodic term considered has disappeared. As l, g, h occur in the arguments of the periodic terms of the expressions for R and for the coordinates, and as the periodic terms, introduced by the change of variables, are small, we can expand each cosine or sine (as in Art. 111) so as to free the arguments from periodic terms. The form of the new disturbing function is similar to that of the old, except that the new g, h now contain the time explicitly; this is due to the fact that the action of the Sun causes the perigee and the node to revolve. We also find the relations between Λ', G', H' and a_0, e_0, γ_0; the equations for the new variables λ', κ, η, Λ', G', H' being canonical, we are ready for the next operation.

We proceed in exactly the same manner. With the relations, just found, between the new L, G, H and the new a, e, γ (that is, between the old Λ', G', H' and the old a_0, e_0, γ_0) we take the non-periodic part of the new R and one of its periodic terms and with these we solve the canonical equations for the new arbitraries (now considered variable), introducing in the same way six other arbitraries. By this operation another periodic term of R is eliminated. Continuing in the same way, we eliminate one of the periodic terms in R at every operation until R is reduced to a non-periodic part only.

Each change of variables may produce new terms in R and may cause a reappearance of terms whose arguments are the same combinations of l, g, h, l', g', h', as those of terms previously eliminated or as that of the term under consideration; in the first case, they will evidently have the same general form as before, that is, they will be of the form $il+i'g+i''h+i'''l+q$; in the latter cases, the new coefficients will be of a higher order than the coefficients of the terms with the same arguments previously eliminated. The series of operations thus continually raises the order of the coefficients of the periodic terms in R. We go on with the operations until these coefficients become sufficiently small to be neglected. Delaunay has continued them until he has found all terms in the longitude correctly to the seventh order inclusive; in addition, some coefficients are calculated to higher orders when slow convergence indicated the necessity of carrying the approximations further.

The number of operations required is very large. Delaunay retains *all* terms in R up to the eighth order inclusive. He first carries out 57 operations, by means of which he eliminates all periodic terms in R which are of an order less than the fourth. The first operation is that outlined in Art. 195 above; then follows the elimination of the terms with arguments l, $2h+2g+2l-2h'-2g'-2l'\pm l$, etc.—those terms whose coefficients are lowered by the integrations being, in general, considered first. The expression for R (in which every term produced by the successive changes of the variables is shown separately) together with the details of these operations occupy the greater part of Vol. I.; the expression for R alone occupies pages 119–256.

Vol. II. opens with the value of R which remains after the 57 operations have been carried out: it now contains no periodic term of an order less than the fourth and the great majority of the terms are of a higher order. He then makes 435 further operations in order to eliminate these remaining terms. In most of these operations it is not necessary to change the variables in R: the small changes produced are made in the coordinates only. There are, however, five periodic terms, arising from changes in R, to be taken into account and these are eliminated by five further operations. Then follow the values of the longitude, latitude and parallax with the successive modifications, written out in full, which they have undergone owing to the $57+435+5=497$ operations. The next chapter is devoted to the further researches into the longitude necessary to carry some of the coefficients to higher orders; this demands a recalculation of some of the operations. In performing them, certain errors are detected and the necessary corrections are added. Finally he gives the reduced values of the coordinates after the change of arbitraries (explained in Art. 199 below) has been made, together with the numerical value of each term in every coefficient, for the case of the Moon.

198. Finally, the disturbing function is reduced to a non-periodic term $-B$. Since B does not contain l, g, h, the canonical equations give

$$\frac{dL}{dt}=0,\quad \frac{dG}{dt}=0,\quad \frac{dH}{dt}=0;\qquad \frac{dl}{dt}=\frac{\partial B}{\partial L},\quad \frac{dg}{dt}=\frac{\partial B}{\partial G},\quad \frac{dh}{dt}=\frac{\partial B}{\partial H}.$$

Hence L, G, H and therefore a, e, γ are unchanged, while we have for l, g, h, respectively, the values

$$(l)+l_0t,\qquad (g)+g_0t,\qquad (h)+h_0t,$$

where l_0, g_0, h_0 are the values of $\partial B/\partial L$, $\partial B/\partial G$, $\partial B/\partial H$ and (l), (g), (h) are arbitraries. Since the previous operation has furnished the connection between L, G, H and a, e, γ, we can obtain l_0, g_0, h_0 as a function of a, e, γ.

The final expressions for v, U, $1/r$ are therefore obtained as a sum of periodic terms whose arguments are of the general form

$$il + i'g + i''h + i'''l' - i''(g' + h'),$$

and whose coefficients are functions of the constants a, e, γ introduced by the last operation; also, l, g, h are each of the form, $t \times$ *function of* a, e, γ + *const.* Further, v contains the term $t \times$ *function of* a, e, γ + *const.* and $1/r$ contains a constant term which is a function of a, e, γ. We must now see how the final l, g, h, a, e, γ are related to the quantities denoted by those letters in purely elliptic motion.

The Arbitrary Constants and the Mean Motions of the Perigee and the Node.

199. The result of any operation was to replace a by a + periodic terms introduced by the operation: the periodic terms, depending on the action of the Sun, will be small. Similar remarks apply to e, γ. Hence a, e, γ at any stage will differ from their original values (which were arbitraries of the elliptic solution) by terms depending on the action of the Sun, and their principal parts will be their elliptic values.

Again, after any operation we find for l, g or h expressions of the form,

$$\text{Arb. const.} + t \times \text{function of } a,\ e,\ \gamma + \text{periodic terms.}$$

The new l, g, h are the non-periodic parts of these, so that l is replaced by l + periodic terms; similarly for g, h.

At the outset we had $l = nt + \epsilon - \varpi$, $g = \varpi - \theta$, $h = \theta$ (θ being here the longitude of the node). Hence the relations of the final l, g, h to their initial values are, when the whole series of operations is completed,

$$\begin{aligned}
\text{Final } l &= nt + \epsilon - \varpi + t \times \text{function of } a,\ e,\ \gamma,\\
\text{Final } g &= \varpi - \theta \qquad + t \times \text{function of } a,\ e,\ \gamma,\\
\text{Final } h &= \theta \qquad\qquad + t \times \text{function of } a,\ e,\ \gamma.
\end{aligned}$$

The last terms of each of these expressions, depending on the action of the Sun, are all small.

Since a', e', n' are present and since at any stage n is defined by the relation $\mu = n^2a^3$, the coordinate v can be expressed in the form,

Const. + $t \times$ const. + periodic terms with coefficients depending on n'/n, e, e', γ, a/a' and arguments depending on l, l', g, $h - l' - g' - h'$.

The coordinates U, $1/r$ are expressed by periodic terms of similar form, the coordinate $1/r$ having further a constant term. We now change the arbitraries so that they may be defined as in Chap. VIII. and consequently be independent of the method of solution adopted.

200. Since the constant parts of l, g, h are arbitraries, we define ϵ, θ, ϖ in disturbed (or undisturbed) motion as follows:

$\epsilon =$ the constant term in $l+g+h$, that is, the constant part of the mean longitude,

$\varpi =$ the constant term in $g+h$,

$\theta =$ the constant term in h.

Again, since n (or a), e, γ are arbitraries, we take a new n, e, γ, a defined as follows:

$n =$ coefficient of t in $l+g+h$, that is, n is the mean motion in longitude whether we consider disturbed or undisturbed motion;

e is such that the coefficient of $\sin l$ in longitude is the same in disturbed motion as in undisturbed motion;

γ is such that the coefficient of $\sin(l+g)$ in latitude is the same as in undisturbed motion;

$a = (\mu/n^2)^{\frac{1}{3}}$, where n has the meaning just defined.

In order to transform to these new arbitraries we equate n to the coefficient of t in the final non-periodic part of v; e, γ are found by equating the coefficients of the principal elliptic term in longitude and the principal term in latitude, found from purely elliptic motion (with e, γ as the eccentricity and the sine of half the inclination, respectively), to the coefficients of the corresponding terms found by the theory. We have then sufficient equations to express the old arbitraries in terms of the new and thence all the coefficients can be expressed in terms of the new arbitraries.

Since $l+g+h$, l, $l+g$ are respectively the mean longitude, the argument of the principal elliptic term and the argument of the principal term in latitude, the mean motions of the perigee and the node are given by the coefficients of t in the final expressions for $l+g+h-l=g+h$ and $l+g+h-l-g=h$, respectively; these coefficients of t must also be expressed in terms of the new a, e, γ, n. They were denoted in de Pontécoulant's theory by $(1-c)n$, $(1-g)n$.

The arguments of all terms are combinations of the four angles $l, l', l+g, l+g+h-l'-g'-h'$. Delaunay puts $D=l+g+h-l'-g'-h'$, $F=l+g$, so that

$$
\begin{aligned}
2D &= \text{Argument of the Variation,}\\
l &= \quad ,, \quad ,, \quad \text{Principal Elliptic term,}\\
l' &= \quad ,, \quad ,, \quad \text{Annual Equation,}\\
D &= \quad ,, \quad ,, \quad \text{Parallactic Inequality,}\\
F &= \quad ,, \quad ,, \quad \text{Principal term in Latitude.}
\end{aligned}
$$

These were respectively denoted by 2ξ, ϕ, ϕ', ξ, η in de Pontécoulant's theory. It is necessary, in Delaunay's final results, to replace a/a' by

$$(E-M)\,a/(E+M)\,a',$$

in order to take into account the correction obtained in Art. 7.

201. The literal results obtained by Delaunay in using the methods explained above, far surpass any other complete developments in their general accuracy and the high order of approximation to which they are carried, although further terms of certain portions, such as the principal parts of the mean motions of the perigee and the node, have been found to a greater degree of approximation. The only results which can be at all compared with them are those of Hansen. The latter, however, confined his attention to numerical developments by substituting the values of m, e, γ, e', a/a' at the outset, while Delaunay gives complete literal results for the three coordinates—this being necessary in his method of treatment. Although the disturbances produced by the Sun are alone treated, the method can be and indeed has been continued from the point where Delaunay stopped, so as to include the effects produced by the actions of the planets, the figure of the Earth, etc. (see Chap. XIII.). Had Delaunay lived, it was his intention to complete the lunar theory by a full examination of all these inequalities and so add a third volume to the two large ones already referred to.

M. Tisserand graphically described Delaunay's work in the following terms* :—'Cette 'théorie est très intéressante au point du vue analytique ; dans la pratique, elle atteint le 'but poursuivi, mais au prix de calculs algébriques effrayants. C'est comme une machine 'aux rouages savamment combinés qu'on appliquerait presque indéfiniment pour broyer un 'obstacle, fragments par fragments. On ne saurait trop admirer la patience de l'auteur, 'qui a consacré plus de vingt années de sa vie à l'exécution matérielle des calculs algébriques 'qu'il a effectués tout *seul*.'

* *Méc. Cél.* vol. III. p. 232.

CHAPTER X.

THE METHOD OF HANSEN.

202. THIS chapter contains an explanation of the methods adopted by Hansen to solve the lunar problem. In the earlier portion of the chapter—to the end of Art. 223—the various equations to be used are formed in a perfectly general manner; the next portion—from Arts. 224 to 238—contains an explanation of the manner in which the approximations, as far as the first order of the disturbing forces, are carried out. When these have been grasped, the extensions necessary for the further approximations follow very easily; they will be outlined in Arts. 239, 240.

For convenience of reference, the notation is based on that of the Darlegung*; in the few places where a different notation is adopted in order to avoid confusion, the differences will be pointed out.

The distinguishing features of Hansen's method are: (i) the angular perturbations in the plane of the orbit are added to the *mean* anomaly of an auxiliary ellipse placed in the plane of the instantaneous orbit, its major axis and eccentricity being constant and its perigee moving in a given manner; (ii) the radial perturbations are determined by finding the ratio of the actual radius vector to the portion of it cut off by the auxiliary ellipse†; (iii) the reckoning of longitudes from a departure point (Art. 79) in the plane of the orbit; (iv) the discovery and use of *one* function W to find all the inequalities in the plane of the orbit; (v) the perfect generality of the method which permits, without difficulty, the inclusion of inequalities from every source; (vi) the completeness with which the method is worked out numerically and the close agreement with observation of the tables which were founded on the theory.

* This title refers to Hansen's paper entitled *Darlegung der theoretischen Berechnung der in den Mondtafeln angewandten Störungen.* Abh. der. K. Sächs. Gesell. d. Wissenschaften, Vol. VI. pp. 91–498, Vol. VII. pp. 1–399. The two parts will be referred to as I., II.

† In the *Fundamenta* (see footnote, p. 36) Hansen finds the logarithm of this ratio.

A general explanation of Hansen's method has been given in a note by Delaunay* and also in two papers by Hansen, *Note sur la théorie des perturbations planétaires* and *Bemerkungen über die Behandlung der Theorie der Störungen des Mondes*†.

203. Hansen's theory is much the most difficult to understand of any of those given up to the present time, partly on account of the somewhat uncouth form in which it is given in the *Fundamenta* and partly on account of the very unusual way in which the perturbations are expressed. It was first published in a series of papers entitled, *Disquisitiones circa theoriam perturbationum quae motum corporum coelestium afficiunt* and *Commentatio de corporum coelestium perturbationibus*‡. The methods, although they are in general there worked out with special reference to the planetary theory only, are, after a few changes, equally applicable to the lunar theory: the chief difference being that, in the former, terms increasing with the time are permitted to be present while, in the latter, they are eliminated by the introduction of a certain quantity y. In the *Fundamenta*, which was published in 1838, the methods, as far as they refer to the Moon's motion, are fully elaborated and detailed expansions are given in forms ready for calculation. A method for the solution, on the same lines, of the problem of four bodies is added.

In 1857 Hansen began another series of papers§ in which the perturbations are expressed in a similar manner, but the methods of arriving at the equations are much simpler. The first paper|| refers to the planetary theory: the method is the same as in the *Darlegung* which followed a few years later. The latter was chiefly published in order to verify the 'Tables de la Lune'¶ which had been previously formed by an application of the principles explained in the *Fundamenta*. As far as p. 212 of the first part, the *Darlegung* is, however, available for the general development and it will be used for that purpose here; when it is a question of forming the successive approximations, the *Fundamenta* must be referred to. The early parts of the *Fundamenta* and of the *Darlegung*, though expressed in forms very different in appearance, can, with some trouble, be seen to be equivalent.

204. *Change of Notation.*

In order that the expressions obtained below may be the same as those of Hansen, a few changes from the notation of Chaps. I—VIII. are necessary; these changes chiefly affect the results of Art. 82, which will be required directly.

Replace $\mathfrak{P},\quad \mathfrak{T},\quad \mathfrak{Z},\quad R$

by $\mu\mathfrak{P},\quad \mu\mathfrak{T},\quad \mu\mathfrak{Z},\quad \mu R,$

respectively. The right-hand members of the equations of Art. 82 must therefore be all multiplied by μ; the results of Art. 75 will remain unaltered. Also, in Art. 124, we have put $R/\mu = R^{(1)} + R^{(2)} + \ldots$; we shall have now

$$R = R^{(1)} + R^{(2)} + \ldots,$$

* *Jour. des Savants*, 1858, pp. 16, 17.

† *Astr. Nach.* Vol. XV. Cols. 201–216, Vol. XIX. Cols. 33–92.

‡ These are contained in various numbers of the *Astr. Nach.* from 1829–1836.

§ These were published in the early volumes of the *Abh. d. K. Sächs. Ges. der Wissensch.*

|| *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten.* Abh. Vol. v. pp. 1–148.

¶ London, 1857. Published by the Government.

where $R^{(1)} + R^{(2)} + \ldots$ retain the same meanings as before*. The mean anomalies w, w' will be replaced later by g, g', respectively.

Hansen also uses h in a different sense. He puts

$$h = \frac{na}{\sqrt{1-e^2}} = \sqrt{\left(\frac{\mu}{l}\right)}, \quad \text{giving} \quad na^2\sqrt{1-e^2} = \frac{\mu}{h} \quad \ldots\ldots\ldots\ldots (1).$$

The significations of the other quantities present in the equations of Art. 82, remain unaltered.

The instantaneous elliptic orbit.

205. The elements of the instantaneous orbit being denoted by a, n, e, ϵ, ϖ, θ, i, the disturbing forces by $\mu\mathfrak{P}$, $\mu\mathfrak{T}$, $\mu\mathfrak{Z}$, the true anomaly by f, the radius vector by r, the latus rectum by l, the distance of the Moon from the node by L, we have from the second, third, fifth and sixth of equations (16), Art. 82, after replacing therein $\mathfrak{P}$, $\mathfrak{T}$, $\mathfrak{Z}$ by $\mu\mathfrak{P}$, $\mu\mathfrak{T}$, $\mu\mathfrak{Z}$ and using the expression for h just given,

$$\left.\begin{aligned} \frac{de}{dt} &= \frac{\mu}{h}\left\{\mathfrak{P}\sin f + \mathfrak{T}\left(\frac{l}{er} - \frac{r}{ea}\right)\right\}, \\ \frac{d\varpi}{dt} - (1-\cos i)\frac{d\theta}{dt} &= \frac{\mu}{eh}\left\{-\mathfrak{P}\cos f + \mathfrak{T}\left(1 + \frac{r}{l}\right)\sin f\right\}, \\ \sin i\,\frac{d\theta}{dt} = h\mathfrak{Z}r\sin L, &\qquad \frac{di}{dt} = h\mathfrak{Z}r\cos L \end{aligned}\right\} \ldots\ldots\ldots (2).$$

We also have from Art. 77, after putting $\mu\mathfrak{T}$ for $\mathfrak{T}$,

$$\mu\mathfrak{T}r = d\,(na^2\sqrt{1-e^2})/dt;$$

whence by (1),

$$\frac{dh}{dt} = -h^2\mathfrak{T}r = -\frac{\mu}{l}\mathfrak{T}r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The last equation replaces that for da/dt in Art. 82. The equation for $d\epsilon_1/dt$ will not be required since those functions of the instantaneous elements, which are used in Hansen's particular method of treatment, do not directly involve ϵ. The method being to find the perturbations of the mean anomaly, the equation which would be obtained by making ϵ vary, is really included in the equation giving the disturbed value of the mean anomaly.

206. In Hansen's method the plane of the Sun's orbit is not necessarily a fixed one. We take as a fixed plane of reference either the ecliptic at a given date or the Invariable plane (Art. 28); any fixed plane inclined at a small angle to the ecliptic will serve at present. As before, we define the positions of all lines by means of their intersections with the unit sphere. Let x be a fixed point on $x\Omega_1$—the plane of reference.

* Hansen uses Ω instead of R.

Let X be a departure point (Art. 79) on $X\Omega_1$—the instantaneous orbit of the Moon. Let Ω_1 be the node of the instantaneous orbit with the fixed

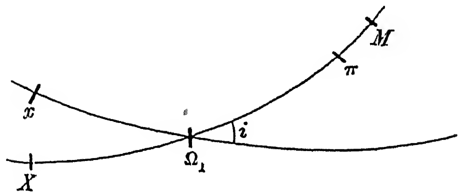

Fig. 8.

plane, π the instantaneous position of its perigee and M the corresponding position of the Moon's radius vector.

We have $$x\Omega_1 = \theta, \quad \Omega_1\pi = \varpi - \theta, \quad \Omega_1 M = L.$$

Let $$X\Omega_1 = \sigma, \quad X\pi = \chi, \quad XM = v = f + \chi;$$

then $$L = v - \sigma, \quad \varpi - \theta = \chi - \sigma.$$

Also, from Art. 101,
$$d\sigma = \cos i \, d\theta,$$
and therefore
$$\frac{d\chi}{dt} = \frac{d\varpi}{dt} - \frac{d\theta}{dt} + \frac{d\sigma}{dt} = \frac{d\varpi}{dt} - (1 - \cos i)\frac{d\theta}{dt}.$$

Whence, the second of equations (2) gives
$$he\frac{d\chi}{dt} = \mu\left\{-\mathfrak{P}\cos f + \mathfrak{T}\left(1 + \frac{r}{l}\right)\sin f\right\} \quad\text{.................(4)};$$
the third and fourth of the same equations become
$$\sin i\frac{d\theta}{dt} = h\mathfrak{Z}r\sin(v - \sigma), \quad \frac{di}{dt} = h\mathfrak{Z}r\cos(v - \sigma) \quad\text{........... (5)}.$$

These three, with the equations for de/dt, dh/dt, given in the previous article, are all we shall require.

The new element χ, like σ, is a pseudo-element and its presence is due to the measuring of the coordinate v* from a departure point. It is not a complete substitute for ϖ since the point X is not completely definite; in order to make it so, it is necessary to define the initial position of X. The latter is assumed to be such that when $i = 0$, X coincides with x; hence X is on that orthogonal to the orbit which passes through x (Art. 79).

The equations for de/dt, dh/dt give
$$\frac{d}{dt}(he) = h\frac{de}{dt} + e\frac{dh}{dt}$$
$$= \mu\left\{\mathfrak{P}\sin f + \mathfrak{T}\left(\frac{l}{er} - \frac{r}{ea} - \frac{er}{l}\right)\right\}.$$

* See Art. 101, where this coordinate is called v_1. In the *Fundamenta*, p. 37, it is denoted by v, and in the *Darlegung*, I. p. 102 by v.

11—2

But since $l/r = 1 + e\cos f$, we have

$$\frac{l}{r} - \frac{r}{a} - e^2\frac{r}{l} = \frac{l}{r} - \frac{r}{l} = \left(1 + \frac{r}{l}\right) e\cos f,$$

and therefore

$$\frac{d}{dt}(he) = \mu\left\{\mathfrak{P}\sin f + \mathfrak{T}\left(1 + \frac{r}{l}\right)\cos f\right\} \quad\text{.................. (6).}$$

Let β_1 be any function of t. Multiply (6) by $\cos(\chi - \beta_1)$ and (4) by $\sin(\chi - \beta_1)$ and subtract: we obtain

$$\frac{d}{dt}\{he\cos(\chi - \beta_1)\} = \mu\left\{\mathfrak{P}\sin(f + \chi - \beta_1) + \mathfrak{T}\left(1 + \frac{r}{l}\right)\cos(f + \chi - \beta_1)\right\}$$
$$+ he\frac{d\beta_1}{dt}\sin(\chi - \beta_1)\text{...(7):}$$

the expression for that function of the instantaneous elements required by Hansen.

207. The considerations which guided Hansen in his method of treating perturbations are set forth in a reply* to some ill-founded criticisms by de Pontécoulant on the *Fundamenta*. Hansen remarks that the solution of the equations which give the elements in terms of the time is very troublesome, requiring that six integrations be performed. But the quantities really sought are not the variable values of the elements but only *three* definite functions of them, namely, the three coordinates. He therefore sought for functions of these elements, by means of which the coordinates could be found in a more direct manner. It is true that, in any case, six integrations must be performed and also that some method of continued approximation must be used, but the ease or difficulty of carrying them out varies enormously according to the plan of treatment. The most numerous of the inequalities in the Moon's motion are those which occur in the plane of the orbit. Hansen succeeded in obtaining a function W, the equation for which was of the first degree; when this function is known in terms of the time, two very simple integrations furnish the inequalities in the plane of the orbit.

One point which differentiates Hansen's methods from all others consists in the addition of the perturbations directly to the *mean* anomaly of a certain auxiliary ellipse in the plane of the instantaneous orbit instead of to the *true* anomaly or to the true longitude on the fixed plane. This fact is sometimes stated by saying that he uses a variable time. The auxiliary ellipse will now be defined: it may be looked upon as the intermediate orbit adopted by Hansen.

The Auxiliary Ellipse.

208. Consider an auxiliary ellipse placed in the plane of the Moon's instantaneous orbit, with one focus at the origin. Let its mean anomaly be denoted by $n_0 z$, its major axis by $2a_0$, where $n_0^2 a_0^3 = \mu$ (μ = sum of the masses of the Earth and the Moon), and its eccentricity by e_0. Throughout the whole of the theory n_0, a_0, e_0 are *absolutely constant.*

* 'Note sur la théorie des perturbations planétaires.' *Astr. Nach.* Vol. xv. Cols. 201–216.

In the auxiliary ellipse, let $\bar{E}$ be the eccentric anomaly*, $\bar{f}$ the true anomaly, $\bar{r}$ the radius vector. We have then (Art. 32)

$$\left.\begin{array}{ll}\bar{r}\cos\bar{f}=a_0(\cos\bar{E}-e_0), & \bar{E}-e_0\sin\bar{E}=n_0 z,\\ \bar{r}\sin\bar{f}=a_0\sqrt{1-e_0^2}\sin\bar{E}, & n_0^2a_0^3=\mu\end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots(8);$$

these, after the elimination of $\bar{E}$, will give $\bar{r}$, $\bar{f}$ as functions of one variable z and of the constants a_0, n_0, e_0.

Also let

$$l_0=a_0(1-e_0^2),\qquad h_0=\frac{n_0a_0}{\sqrt{1-e_0^2}}=\sqrt{\left(\frac{\mu}{l_0}\right)},\qquad n_0a_0^2\sqrt{1-e_0^2}=\frac{\mu}{h_0}\ \ldots\ldots(9);$$

so that h_0 is the same function of n_0, a_0, e_0 that h was of n, a, e.

Let the perigee of this ellipse have a forward motion in the plane of the orbit equal to $n_0 y$, and let π_0 be the longitude of the perigee from the departure point X at time $t=0$. The longitude from X of the point whose true anomaly is $\bar{f}$, will be at time t,

$$\bar{f}+n_0yt+\pi_0.$$

This ellipse being used as an intermediate orbit, we shall have initially, $z=t+$ const. or $n_0z=g$. The plane of the orbit is then supposed to be fixed and X will be a fixed point on it. Also n_0, $2a_0$, e_0 will be the mean motion, major axis and eccentricity, while y is a constant, as yet indeterminate, depending on the Sun's action in the same manner as did the constant c introduced in Chap. IV. When the complete action of the Sun is taken into account, the value of n_0z will be $g+$ periodic terms.

209. So far the only relation between the auxiliary ellipse and the actual position of the Moon consists in the fact that the former is placed in the instantaneous plane of the orbit. The connecting link is made by allowing the point whose true anomaly is $\bar{f}$ to be on the actual radius vector of the Moon. This fact, expressed in symbols, is, by Arts. 206, 208,

$$f+\chi=v=\bar{f}+n_0yt+\pi_0,$$

so that $\bar{r}$, $\bar{f}$ are the radius vector and true anomaly of the point on the auxiliary ellipse where the actual radius vector of the Moon cuts this ellipse. When z and the actual position of the Moon are known in terms of the time and of the constants, the auxiliary ellipse is completely defined.

Let the actual radius vector of the Moon be, as earlier, r and put

$$r=\bar{r}(1+\nu).$$

When z, ν are known in terms of the time and of the constants, the position of the Moon in its orbit will be known. The problem of motion in the instantaneous plane therefore consists in the determination of z, ν and in

* Hansen in the *Darlegung*, I. p. 102, where these equations are given, denotes the eccentric anomaly by ϵ.

the determination of the meanings to be attached to the constants n_0, a_0, e_0 and to the arbitraries which arise when the equations for z, ν are integrated.

210. Some idea of Hansen's method can now be given. Suppose that the initial position of X has been defined and that z, ν have been expressed in terms of the time. The equations (8) will then give $\bar{r}, \bar{f}$; from these, by means of the equations

$$v = \bar{f} + n_0 yt + \pi_0, \qquad r = \bar{r}(1 + \nu),$$

we can calculate v, r; y is a certain quantity (which is constant when the solar perturbations only are considered) to be defined during the process of solution so that no terms, increasing continually in proportion with the time, shall be present in the expressions for $n_0 z - g$, ν. The first object to be sought is therefore the determination of z, ν.

The second object in view is the determination of the motion of the plane of the orbit; this is given by the equations (5). But in order to reduce the longitude in the orbit to that along the plane of reference we must know σ. The latter is found, when i, θ are known in terms of the time, from the equation

$$\frac{d\sigma}{dt} = \cos i \frac{d\theta}{dt}.$$

Also, when σ, i, θ are known, the latitude above the plane of reference will be obtainable.

The determination of z, ν will be reduced to the consideration of a function W which will presently be constructed; the variables σ, i, θ will be replaced by three others. The integration of the equations for W, z, ν will furnish four arbitrary constants which will be determined in Art. 231, and those for the variables σ, i, θ three further arbitrary constants; the latter three will furnish the initial position of X and of the plane of the orbit. All the equations considered are reduced to the first order. The equations for z, ν are really of the second order, since W is determined by an equation of the first order. The equations for P, Q, K—the variables which ultimately replace σ, i, θ—are each of the first order, so that the seven arbitrary constants are *necessary* for the general solution of the equations. Six constants only are necessary to define the position of the Moon: the seventh constant is that which defines the initial position of X.

The Equations for z, ν.

211. Since v, the longitude in the orbit, is measured from a departure point, dv/dt has the same form, when expressed in terms of the instantaneous elements and of the time, in disturbed or in undisturbed motion (Art. 79); hence $r^2 dv/dt = na^2\sqrt{1-e^2}$, in disturbed motion. From this and from the equations of Art. 208 we have, as in Art. 32,

$$\left.\begin{aligned}
&\frac{a(1-e^2)}{r} = 1 + e\cos f, && \frac{a_0(1-e_0^2)}{\bar{r}} = 1 + e_0 \cos\bar{f}\\
&r^2\frac{dv}{dt} = na^2\sqrt{1-e^2} = \frac{\mu}{h}, && \bar{r}^2\frac{d\bar{f}}{dz} = n_0 a_0^2\sqrt{1-e_0^2} = \frac{\mu}{h_0}\\
&\frac{dr}{dt} = \frac{nae}{\sqrt{1-e^2}}\sin f = he\sin f, && \frac{d\bar{r}}{dz} = \frac{n_0 a_0 e_0}{\sqrt{1-e_0^2}}\sin\bar{f} = h_0 e_0 \sin\bar{f}\\
&\qquad f + \chi = v = \bar{f} + n_0 yt + \pi_0, \\
&\qquad r = \bar{r}(1+\nu).
\end{aligned}\right\} \ldots(10).$$

Of these, it is to be remembered that the portions to the left, involving the letters r, f, a, etc., refer to the instantaneous ellipse; those to the right, involving $\bar{r}$, $\bar{f}$, a_0, etc., refer to the auxiliary ellipse. The connection is furnished by the two values for v and by the fact that the two ellipses lie in the same plane.

Since $\bar{f}$ is a function of one variable z which is itself supposed to be a function of t, we deduce immediately

$$\frac{\mu}{hr^2} = \frac{dv}{dt} = \frac{d\bar{f}}{dz}\frac{dz}{dt} + n_0 y.$$

Eliminating $d\bar{f}/dz$ from this equation by means of the value given for it in (10), we obtain

$$\frac{dz}{dt} = \frac{h_0 \bar{r}^2}{hr^2} - \frac{y}{\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2 \quad\text{........................(11).}$$

Also, from the last of the same equations,

$$\frac{\bar{r}^2}{r^2} = -1 + 2\frac{\bar{r}}{r} + \left(\frac{\nu}{1+\nu}\right)^2 = -1 + 2\frac{h^2}{h_0^2}\frac{\bar{r}}{a_0}\frac{1+e\cos f}{1-e_0^2} + \left(\frac{\nu}{1+\nu}\right)^2,$$

since $h_0^2 l_0 = \mu = h^2 l$. Substituting in (11) and putting $f = \bar{f} + n_0 y + \pi_0 - \chi$, we find

$$\frac{dz}{dt} = 1 + \overline{W} + \frac{h_0}{h}\left(\frac{\nu}{1+\nu}\right)^2 - \frac{y}{\sqrt{(1-e_0^2)}}\left(\frac{\bar{r}}{a_0}\right)^2 \quad\text{.......... ...(12),}$$

where
$$\overline{W} = -1 - \frac{h_0}{h} + 2\frac{h}{h_0}\frac{\bar{r}}{a_0}\frac{1 + e\cos(\bar{f} + n_0 yt + \pi_0 - \chi)}{1-e_0^2} \quad\text{.........(13).}$$

These furnish the required equation for z.

212. To find the equation for ν we have, since $\bar{r}$ is a function of z only,

$$\frac{dr}{dt} = (1+\nu)\frac{d\bar{r}}{dz}\frac{dz}{dt} + \bar{r}\frac{d\nu}{dt}.$$

Substituting from (11) for dz/dt and observing that $1 + \nu = r/\bar{r}$, we obtain

$$\frac{d\nu}{dt} = -(1+\nu)\frac{d\bar{r}}{dz}\frac{h_0\bar{r}}{hr^2} + \frac{1}{\bar{r}}\frac{dr}{dt} + \frac{y(1+\nu)}{a_0\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\frac{d\bar{r}}{dz}$$

$$= -\frac{h_0}{hr}\frac{d\bar{r}}{dz} + \frac{1}{\bar{r}}\frac{dr}{dt} + \tfrac{1}{2}\frac{y(1+\nu)}{\sqrt{1-e_0^2}}\frac{d}{dz}\left(\frac{\bar{r}}{a_0}\right)^2.$$

Put for $\dfrac{1}{r}$, $\dfrac{d\bar{r}}{dz}$, $\dfrac{1}{\bar{r}}$, $\dfrac{dr}{dt}$ their values from (10). The first and second terms of the latter expression for $d\nu/dt$ become

$$-\frac{h_0}{h}\frac{1+e\cos f}{a(1-e^2)}h_0 e_0 \sin\bar{f} + \frac{1+e_0\cos\bar{f}}{a_0(1-e_0^2)}he\sin f;$$

or, since $h_0^2/h^2 = a(1-e^2)/a_0(1-e_0^2)$, they are

$$\frac{h}{a_0(1-e_0^2)}\{-e_0 \sin \bar{f}(1+e\cos f)+e\sin f(1+e_0\cos \bar{f})\} \quad\text{.......(14).}$$

But, differentiating (13) partially with respect to z, we have (since z is only present explicitly in $\bar{r}, \bar{f}$), after inserting the values of $d\bar{r}/dz$, $d\bar{f}/dz$ given in (10),

$$\frac{\partial \overline{W}}{\partial z} = 2\frac{h}{h_0}\frac{h_0 e_0 \sin \bar{f}}{a_0(1-e_0^2)}\{1+e\cos(\bar{f}+n_0 yt+\pi_0-\chi)\}$$
$$-2\frac{h}{h_0}\frac{\bar{r}e}{a_0(1-e_0^2)}\frac{n_0 a_0^2\sqrt{1-e_0^2}}{\bar{r}^2}\sin(\bar{f}+n_0 yt+\pi_0-\chi),$$

which, since $\bar{f}+n_0 yt+\pi_0-\chi = f$, $n_0 a_0 = h_0\sqrt{1-e_0^2}$, becomes

$$\frac{\partial \overline{W}}{\partial z} = \frac{2h}{a_0(1-e_0^2)}\left\{e_0 \sin \bar{f}(1+e\cos f) - \frac{a_0(1-e_0^2)}{\bar{r}}e\sin f\right\}$$
$$= \frac{2h}{a_0(1-e_0^2)}\{e_0\sin\bar{f}(1+e\cos f) \;-(1+e_0\cos\bar{f})\,e\sin f\}.$$

Comparing this with the expression (14) which contains the first two terms of dv/dt, we obtain

$$\frac{dv}{dt} = -\tfrac{1}{2}\frac{\partial \overline{W}}{\partial z}+\tfrac{1}{2}\frac{y(1+v)}{\sqrt{1-e_0^2}}\frac{d}{dz}\left(\frac{\bar{r}}{a_0}\right)^2 \quad\text{..................(15),}$$

the required equation for v.

213. It is easy to see that when the two sets of elements coincide, $\overline{W}$, v, y vanish; further, if the disturbing forces vanish, $dz/dt = 1$. The quantities $\overline{W}$, v, y are therefore at least of the first order of the disturbing forces. Hence, in the expression (12) for dz/dt, the third term is of the order of the square of the disturbing forces and it may be neglected in the first approximation; in the fourth term we can, to the same degree of accuracy, put $n_0 z = n_0 t + \text{const.} = g$: this amounts to neglecting the disturbing forces in the coefficient of y. The same remark may be made concerning the second term in the expression (15) for dv/dt. Hence, the principal parts of the equations for z, v depend on $\overline{W}$ and this function must now be expressed in terms of the disturbing forces.

So far the equations (12), (15) are purely algebraical results obtained by the combination of two sets of elliptic formulæ and connected by the single fact that the longitude in each orbit, reckoned from one origin, is the same. One mean anomaly is therefore a function of the other, but no supposition, involving any relations between the two sets of elements, has been made and the results would be equally true for any two sets of elements in one of which the motion of the perigee is directly proportional to that of the mean anomaly*.

* See Hansen, *Ueber die Anwendung osculirender Elemente als Grundlage der Berechnung der Störungen eines Planeten, und über die unabhängigen Elemente der "Fundamenta nova etc."* Astr. Nach. Vol. XVIII. Cols. 237-288.

The equation for W.

214. The expression (13) for $\overline{W}$ is a function of the variable elements h, e, χ; it also contains t through the term $n_0 yt$ and through $\bar{r}, \bar{f}$—the latter being functions of z and therefore of t. But since the equations which express h, e, χ in terms of the disturbing forces are given by their differentials, it will be better to form $d\overline{W}/dt$ and then to find $\overline{W}$ by a single integration, instead of performing the three integrations necessary to find h, e, χ directly and then substituting their values in the expression for $\overline{W}$. It will now be shown that, *in performing these processes, $\bar{r}$, $\bar{f}$, which are functions of z and therefore of t, may be considered constant.* (See Art. 104.)

Let τ be a constant and let $\zeta, \bar{\rho}, \bar{\phi}$ denote the values of $z, \bar{r}, \bar{f}$ when τ replaces t: $\zeta, \bar{\rho}, \bar{\phi}$ are then the same functions of the constant τ that $z, \bar{r}, \bar{f}$ are of t. Let W denote the value of $\overline{W}$ when $\bar{\rho}, \bar{\phi}$ are put for $\bar{r}, \bar{f}$.

Now the expression (13) for $\overline{W}$ may be put into the form

$$\overline{W} = L_1 + L_2 \bar{r} + L_3 \bar{r} \cos \bar{f} + L_4 \bar{r} \sin \bar{f},$$

where L_1, L_2, L_3, L_4 do not contain $\bar{r}, \bar{f}$, being functions of $h, e, \chi, n_0 yt$ only. We may write this

$$\overline{W} = \int \frac{dL_1}{dt} dt + \bar{r} \int \frac{dL_2}{dt} dt + \bar{r} \cos \bar{f} \int \frac{dL_3}{dt} dt + \bar{r} \sin \bar{f} \int \frac{dL_4}{dt} dt$$

$$= \left[\int \left(\frac{dL_1}{dt} + \bar{\rho} \frac{dL_2}{dt} + \bar{\rho} \cos \bar{\phi} \frac{dL_3}{dt} + \bar{\rho} \sin \bar{\phi} \frac{dL_4}{dt} \right) dt \right]_{\tau = t}$$

$$= \left[\int \frac{dW}{dt} dt \right]_{\tau = t},$$

since τ is a constant. After the integration, τ must be put equal to t. Hence we need only consider the function W, in which $\bar{\rho}, \bar{\phi}$ are constants.

We have, substituting $\bar{\rho}, \bar{\phi}, \zeta$ for $\bar{r}, \bar{f}, z$ respectively, in equations (10),

$$\frac{a_0 (1 - e_0^2)}{\bar{\rho}} = 1 + e_0 \cos \bar{\phi}, \qquad \bar{\rho}^2 \frac{d\bar{\phi}}{d\zeta} = n_0 a_0^2 \sqrt{1 - e_0^2}, \qquad \text{etc.} \quad \ldots (16).$$

215. The definition of W gives

$$W = -1 - \frac{h_0}{h} + 2 \frac{h}{h_0} \frac{\bar{\rho}}{a_0} \frac{1 + e \cos (\bar{\phi} + n_0 yt + \pi_0 - \chi)}{1 - e_0^2} \quad \ldots\ldots\ldots (17).$$

Therefore, considering $\bar{\rho}, \bar{\phi}$ as constant and remembering that $a_0, e_0, n_0, h_0, \pi_0$ are always constant,

$$\frac{dW}{dt} = \frac{h_0}{h^2} \frac{dh}{dt} + \frac{2}{h_0} \frac{\bar{\rho}}{a_0 (1 - e_0^2)} \frac{dh}{dt} + \frac{2}{h_0} \frac{\bar{\rho}}{a_0 (1 - e_0^2)} \frac{d}{dt} \{ he \cos (\chi - \beta_1) \},$$

where $\beta_1 = \bar{\phi} + n_0 yt + \pi_0$.

The equations (3), (7) are immediately applicable. By means of them we find,

$$\frac{dW}{dt} = -h_0\mathfrak{T}r - 2\frac{h^2}{h_0}\frac{\bar\rho}{a_0(1-e_0^2)}\mathfrak{T}r + \frac{2\mu}{h_0}\frac{\bar\rho}{a_0(1-e_0^2)}\left\{\mathfrak{P}\sin(f+\chi-\beta_1)\right.$$
$$\left.+\mathfrak{T}\left(1+\frac{r}{l}\right)\cos(f+\chi-\beta_1)\right\} + 2\frac{h}{h_0}\frac{n_0 y\bar\rho}{a_0(1-e_0^2)}e\sin(\chi-\beta_1).$$

But since $\mu = h^2 a(1-e^2) = h_0^2 a_0(1-e_0^2), \quad l = a(1-e^2),$

$$f+\chi-\beta_1 = \bar f + n_0 yt + \pi_0 - \bar\phi - n_0 yt - \pi_0 = \bar f - \bar\phi,$$

we obtain

$$\frac{dW}{dt} = -h_0\mathfrak{T}r\left(1 + 2\frac{h^2}{h_0^2}\frac{\bar\rho}{a_0(1-e_0^2)}\right) + 2h_0\bar\rho\,\mathfrak{P}\sin(\bar f-\bar\phi) + 2h_0\bar\rho\,\mathfrak{T}\cos(\bar f-\bar\phi)$$
$$+2\frac{h^2}{h_0}\frac{\bar\rho}{a_0(1-e_0^2)}\mathfrak{T}r\cos(\bar f-\bar\phi) - 2\frac{h}{h_0}\frac{n_0 y\bar\rho}{a_0(1-e_0^2)}e\sin(\bar\phi+n_0yt+\pi_0-\chi).$$

In order to get the last term of this expression into a suitable form, differentiate (17) with respect to ζ. This gives

$$\frac{\partial W}{\partial\zeta} = \left(W + \frac{h_0}{h} + 1\right)\frac{1}{\bar\rho}\frac{d\bar\rho}{d\zeta} - 2\frac{h}{h_0}\frac{\bar\rho}{a_0}\frac{d\bar\phi}{d\zeta}\frac{e\sin(\bar\phi+n_0yt+\pi_0-\chi)}{1-e_0^2},$$

which, by means of (16), becomes

$$\frac{\partial W}{\partial\zeta} = \frac{1}{2}\left(W + \frac{h_0}{h} + 1\right)\frac{1}{\bar\rho^2}\frac{d}{d\zeta}\bar\rho^2 - 2\frac{h}{h_0}\frac{n_0 a_0}{\bar\rho\sqrt{1-e_0^2}}e\sin(\bar\phi+n_0yt+\pi_0-\chi).$$

Substituting for $e\sin(\bar\phi+n_0yt+\pi_0-\chi)$ from this result in the last expression for dW/dt and rearranging the other terms, we obtain

$$\frac{dW}{dt} = h_0\mathfrak{T}r\left[2\frac{\bar\rho}{r}\cos(\bar f-\bar\phi) - 1 + 2\frac{h^2}{h_0^2}\frac{\bar\rho}{a_0(1-e_0^2)}\{\cos(\bar f-\bar\phi)-1\}\right]$$
$$+2h_0\frac{\bar\rho}{r}\mathfrak{P}r\sin(\bar f-\bar\phi) + \frac{n_0 y}{\sqrt{1-e_0^2}}\left[\left(\frac{\bar\rho}{a_0}\right)^2\frac{\partial W}{n_0\partial\zeta} - \frac{1}{2}\left(W+\frac{h_0}{h}+1\right)\frac{d}{n_0 d\zeta}\left(\frac{\bar\rho}{a_0}\right)^2\right] \quad (18),$$

which is the required equation for W. When W has been found from this equation and thence, by putting $\tau = t$, the value of $\overline{W}$, the equations (12), (15) will give z, ν. In the process of integrating (18), $\bar\rho$, $\bar\phi$ are, by Art. 214, to be considered constant.

Several methods of arriving at this expression for dW/dt have been given. On pp. 41–43 of the *Fundamenta*, Hansen arrives at it by a direct transformation from the equations of variations of the elements, but the form obtained is slightly different from (18) above; the latter becomes the equation given in the *Darlegung*, I. p. 107, if $\partial R/\partial v$, $\partial R/\partial r$ be substituted for $\mathfrak{T}r$, $\mathfrak{P}$. The method given above is based on one by Zech*. In the *Darlegung* I. pp.

* *Neue Ableitung der Hansen'schen Fundamentalformeln für die Berechnung der Störungen.* Astr. Nach. Vol. XLI. Cols. 129–142, 205–208.

104–107 * is another investigation obtained directly from the fact that the coordinates and the velocities have the same form in disturbed and in undisturbed motion. In all these methods Hansen's theorem, enunciated in Art. 104, is used; Brünnow † gave developments of a different form which suggested that this theorem was not necessary (see *M. N. R. A. S.* 1895, No. 2). Earlier, Cayley ‡ had also given a method of obtaining the equations of the *Fundamenta* which assisted in clearing up several difficult points in that work.

216. Six constants have been introduced with the auxiliary ellipse, namely, a_0, e_0, n_0, y, π_0 and that attached to $n_0 z$ (which in undisturbed motion is of the form $n_0 t + c_0$). These are not all independent and arbitrary. The two a_0, n_0 are connected by the equation $n_0{}^2 a_0{}^3 = \mu$, while y will be seen to be a certain constant necessary (like c in Chaps. IV. VII.) to put the solution into a suitable form. The number of independent constants is therefore four; the other three arise from σ, i, θ (or from P, Q, K). Hence, as far as the motion *in* the plane of the orbit is concerned, we have the necessary number (Art. 210). The four new constants, which will be introduced when the equations for z, ν are integrated, can be determined at will, and they will be so determined that the meanings of n_0, e_0, π_0, c_0 in disturbed motion may be rendered independent of the method of solution adopted. As numerical values are used by Hansen, it is necessary to know beforehand what significations are to be attached to n_0, e_0, h_0. These are, however, better explained when the equations for z, ν have been integrated: the definitions will be found in Art. 231 below. It is only necessary to state here that h_0, e_0 differ from h, e by quantities of the order of the disturbing forces.

It will therefore be seen that the elements with suffix zero are not the purely elliptic values of the instantaneous elements, if we understand by 'purely elliptic values' those with which we start. On any development with the latter as a basis, the observed mean motion, for example, would no longer be denoted by a single letter but would consist of the purely elliptic value together with a series of constant terms due to the disturbing forces. This was seen in de Pontécoulant's theory where the new arbitraries arising during the integrations were used in such a way that the mean motion might be denoted by n. The same thing occurred in Delaunay's theory, but there it was necessary to make a direct transformation in the final results. Hansen, like de Pontécoulant, keeps the arbitraries (denoted by b, ξ below §) which arise in the integrations, for the purpose of defining n_0, e_0. These remarks are necessary for a clear understanding of the three sets of elements used in the *Fundamenta*. There Hansen denotes by (a), (n), etc., the quantities denoted by a_0, n_0, etc., above and by a_0, n_0, etc., the purely elliptic or initial values of a, n, etc. (the latter being the instantaneous elements), that is, the values of a, n, etc., when the disturbing forces vanish ‖. With the notation used in this chapter, and in the *Darlegung* ¶, a_0, n_0, etc. implicitly contain terms due to the disturbing forces.

* It was also given by Hansen in the Astr. Nach. Vol. LXII. Cols. 273–280, *Neue Ableitung meiner Fundamentalformeln für die Berechnung der Störungen.*

† *Saturn etc., nebst einer Ableitung der Hansen'schen Fundamentalformeln.* Astr. Nach. Vol. LXIV. Cols. 259–266.

‡ *On Hansen's Lunar Theory.* Quar. Math. Jour. Vol. I. pp. 112–125; *A Memoir on the Problem of Disturbed Elliptic Motion.* Mem. R. A. S. Vol. XXVII. pp. 1–29; *A Supplementary Memoir on the Problem of Disturbed Elliptic Motion.* Mem. R. A. S. Vol. XXVIII. pp. 217–234. These are also found in his collected works Vol. III. pp. 13–24, 270–292, 344–359.

§ Art. 230.

‖ *Fundamenta*, pp. 62, 64.

¶ *Darlegung*, I. p. 102.

It is necessary to point out that $n_0 y$ is not the mean motion of the Moon's perigee along the true ecliptic although it accounts for the greater part of this motion. It is the mean motion of the perigee in the orbit. A small correction, which depends on the mean motion of the Moon's node along the ecliptic and on the square of the inclination, has to be applied in order to obtain the mean motion of the perigee along the ecliptic. See Arts. 217, 237.

One great advantage of Hansen's method of computing the longitude in the plane of the orbit is that the inequalities produced in z by the motion of the plane of the orbit are necessarily very small. Since the force $\mathfrak{Z}$ does not occur in the equations for z, ν, the inequalities produced in these variables by the motion of the plane of the orbit must all be small quantities of the order of the square of the disturbing forces at least.

The Motion of the Plane of the Moon's orbit.

217. *Definitions.* It is necessary now to define the variables by means of which the motion of the plane of the Moon's instantaneous orbit is found. We suppose here that the Sun's orbit is not fixed but that it is moving in a *known* manner.

On the unit sphere, let $X\Omega M$, $X'\Omega m'$ be the orbits of the Moon and the Sun respectively. Let X' be a departure point on the Sun's orbit, defined in the same manner as X was. Let Ω_1, Ω_1' be the ascending nodes of the orbits on the fixed plane of reference. With the notation used in Art. 206 we have, if accented letters refer to the Sun's orbit, the following old and new definitions*:

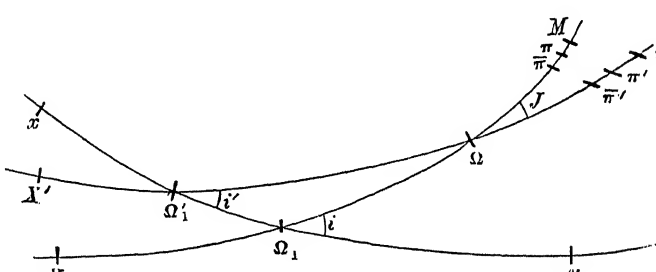

Fig. 9.

$$x\Omega_1 = \theta, \quad y\Omega_1\Omega = i, \quad X\Omega_1 = \sigma, \quad XM = v, \quad X\Omega = \psi,$$

$$x\Omega_1' = \theta', \quad y\Omega_1'\Omega = i', \quad X'\Omega_1' = \sigma', \quad X'm' = v', \quad X'\Omega = \psi',$$

$$X'\Omega X = J.$$

* The angles denoted here by ψ, ψ' are called by Hansen ϕ, ψ respectively. *Fundamenta*, p. 84. *Darlegung*, I. p. 110. The change is made to prevent confusion with the letter ϕ used earlier.

Hence (Art. 101)

$$d\sigma = \cos i \, d\theta, \qquad d\sigma' = \cos i' \, d\theta'.$$

Also, put

$$\left.\begin{array}{ll} p = \sin i \sin \sigma, & p' = \sin i' \sin \sigma', \\ q = \sin i \cos \sigma, & q' = \sin i' \cos \sigma' \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (19).$$

All the quantities denoted by accented letters, except ψ', are supposed known, since they refer solely to the Sun's orbit.

Let N, K be defined by the equations

$$\left.\begin{array}{l} 2N = \pi_0 + \pi_0' - \psi - \psi' - 2n_0\, \alpha t, \\ 2K = \pi_0 - \pi_0' - \psi + \psi' + 2n_0\, \eta t \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

where π_0' denotes the distance of the Sun's apse from X' at time $t = 0$: α, η will be so defined that N, K contain no terms directly proportional to the time.

We have

$$N + K = \pi_0 - \psi - n_0(\alpha - \eta)\, t, \quad N - K = \pi_0' - \psi' - n_0(\alpha + \eta)\, t \ldots (21).$$

Since $X'\Omega = \psi'$, and since N, K contain no terms directly proportional to the time, the quantity $-n_0(\alpha + \eta)$ represents the mean motion of the argument ψ', that is, $-n_0(\alpha + \eta)$ *is the mean motion of the Moon's node along the true ecliptic.*

Again, if $\bar{\pi}$ be the perigee of the auxiliary ellipse, the mean motion of the Moon's perigee, when reckoned along the true ecliptic and then along the orbit, will be the same as that of $\bar{\pi}$, when reckoned in the same way. Now

$$X'\Omega + \Omega\bar{\pi} = X'\Omega + X\bar{\pi} - X\Omega = \psi' + n_0 y t + \pi_0 - \psi,$$

by Art. 208. The mean motion of $\psi' - \psi$ is $-2n_0\eta$, by the second of equations (20). Hence, *the mean motion of the Moon's perigee along the true ecliptic is* $n_0(y - 2\eta)$.

In general, y, α, η are constant quantities. The actions of the planets, however, produce small accelerations in the mean motions of the perigee and of the node, that is, they produce terms dependent on t^2, t^3,... These can be taken into account by putting for $n_0 yt$, $n_0 \alpha t$, $n_0 \eta t$ the integrals $n_0 \int y\, dt$, $n_0 \int \alpha\, dt$, $n_0 \int \eta\, dt$, respectively. The differentials of these quantities with respect to the time will then be still denoted by $n_0 y$, $n_0 \alpha$, $n_0 \eta$, respectively*.

Hansen introduces the quantities Φ, Ψ to denote the angles $\psi - \sigma$, $\psi' - \sigma'$†. They are, however, merely intermediaries in his development of the equations obtained below: as they are not necessary in the proof given here, they will not be used in this sense. He uses the letter Ψ, in another place, to denote an entirely different quantity. See Art. 230 below.

* *Fundamenta*, pp. 51, 97, 98; *Darlegung*, I. pp. 103, 112.

† *Fundamenta*, p. 82; *Darlegung*, I. p. 112.

The equations satisfied by P, Q, K.

218. In any spherical triangle ABC whose sides, denoted by a, b, c and angles, denoted by A, B, C, all vary, we have

$$dC = - dA \cos b - dB \cos a + dc \sin A \sin b\ ,$$

$$db = \ \ dc \cos A + da \cos C + dB \sin C \sin a,$$

$$da = \ \ dc \cos B + db \cos C + dA \sin C \sin b.$$

To prove these, draw BD, AD' so that the angles CBD, CAD' are each equal to a right angle. We have, in the triangle ABC,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \ldots\ldots\ldots(22),$$

and therefore, when the sides and angles all vary,

$$-dC \sin C = dA\,(\sin A \cos B + \cos A \sin B \cos c)$$
$$+dB\,(\cos A \sin B + \sin A \cos B \cos c) - dc \sin A \sin B \sin c.$$

The coefficient of dA in this equation is equal to

$$-\cos(90 - A)\cos(180 - B) + \sin(90 - A)\sin(180 - B)\cos c$$
$$= \cos AD'B = \sin C \cos b,$$

by the spherical triangles ABD', $AD'C$.

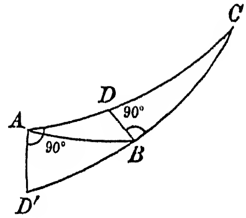

Fig. 10.

Similarly, by considering the triangles ADB, CDB, we prove that the coefficient of dB is equal to $\sin C \cos a$. Also, since $\sin B \sin c = \sin b \sin C$, the coefficient of $-dc$ is equal to $\sin A \sin b \sin C$. Substituting and dividing by $-\sin C$ we obtain the expression given above for dC.

Again, in the result for dC, put $\pi - A$, $\pi - B$, $\pi - C$ for a, b, c and $\pi - a$, $\pi - b$, $\pi - c$ for A, B, C, respectively. We immediately deduce, from the known property of the polar triangle, the value of dc in terms of da, db dC, and thence, by interchanging the letters, the values of db, da, given above.

For the triangle $\Omega'_1\Omega_1\Omega$ (fig. 9), put $A = i'$, $B = 180° - i$, $C = J$, $a = \psi - \sigma$, $b = \psi' - \sigma'$, $c = \theta - \theta'$. We obtain

$$dJ = - di' \cos(\psi' - \sigma') + di \cos(\psi - \sigma) + (d\theta - d\theta') \sin i' \sin(\psi' - \sigma') \ldots (23),$$

$$d\psi' - d\sigma' = \ \ (d\theta - d\theta') \cos i' + (d\psi - d\sigma) \cos J - di \sin J \sin(\psi - \sigma),$$

$$d\psi - d\sigma = -(d\theta - d\theta') \cos i + (d\psi' - d\sigma') \cos J + di' \sin J \sin(\psi' - \sigma').$$

Substituting for $d\sigma$, $d\sigma'$ their values $\cos i\, d\theta$, $\cos i'\, d\theta'$, and transposing, the second and third of these equations become,

$$d\psi' - d\psi \cos J = d\theta\,(\cos i' - \cos i \cos J) - di \sin J \sin(\psi - \sigma)$$
$$= \ \ d\theta \sin i \sin J \cos(\psi - \sigma) - di \sin J \sin(\psi - \sigma),$$

$$d\psi - d\psi' \cos J = d\theta'(\cos i - \cos i' \cos J) + di' \sin J \sin(\psi' - \sigma')$$
$$= - d\theta' \sin i' \sin J \cos(\psi' - \sigma') + di' \sin J \sin(\psi' - \sigma'):$$

the second line in each case being obtained by the successive application of the formula (22) to the triangle $\Omega'_1\Omega_1\Omega$.

But we have, from (20),

$$(d\psi + d\psi')(1 - \cos J) = -4(dN + n_0\alpha\, dt)\sin^2 \tfrac{1}{2}J,$$
$$(d\psi - d\psi')(1 + \cos J) = -4(dK - n_0\eta\, dt)\cos^2 \tfrac{1}{2}J.$$

Therefore, substituting these values in the sum and difference of the two previous equations, we obtain

$$\left.\begin{aligned} dN + n_0\alpha\, dt &= \tfrac{1}{2}\cot\tfrac{1}{2}J\,\{di\sin(\psi-\sigma) - d\theta\sin i\cos(\psi-\sigma) \\ &\qquad - di'\sin(\psi'-\sigma') + d\theta'\sin i'\cos(\psi'-\sigma')\}, \\ dK - n_0\eta\, dt &= \tfrac{1}{2}\tan\tfrac{1}{2}J\,\{-di\sin(\psi-\sigma) + d\theta\sin i\cos(\psi-\sigma) \\ &\qquad - di'\sin(\psi'-\sigma') + d\theta'\sin i'\cos(\psi'-\sigma')\} \end{aligned}\right\} \quad (24).$$

The equations (23), (24) for dJ, dN, dK are purely geometrical results; it is necessary now to introduce the disturbing forces.

We deduce immediately from equations (5),

$$\frac{di}{dt}\cos(\psi-\sigma) + \frac{d\theta}{dt}\sin i\sin(\psi-\sigma) = h\mathfrak{Z}r\cos(v-\psi),$$
$$\frac{di}{dt}\sin(\psi-\sigma) - \frac{d\theta}{dt}\sin i\cos(\psi-\sigma) = -h\mathfrak{Z}r\sin(v-\psi).$$

Also, differentiating the expressions for p', q' in (19) and remembering that $d\sigma' = \cos i'\, d\theta'$, we obtain

$$dp' = di'\cos i'\sin\sigma' + d\theta'\cos i'\sin i'\cos\sigma',$$
$$dq' = di'\cos i'\cos\sigma' - d\theta'\cos i'\sin i'\sin\sigma'.$$

Whence

$$\frac{di'}{dt}\cos(\psi'-\sigma') + \frac{d\theta'}{dt}\sin i'\sin(\psi'-\sigma') = \frac{1}{\cos i'}\left(\frac{dp'}{dt}\sin\psi' + \frac{dq'}{dt}\cos\psi'\right),$$
$$\frac{di'}{dt}\sin(\psi'-\sigma') - \frac{d\theta'}{dt}\sin i'\cos(\psi'-\sigma') = \frac{1}{\cos i'}\left(-\frac{dp'}{dt}\cos\psi' + \frac{dq'}{dt}\sin\psi'\right).$$

Dividing the equations (23), (24) by dt and using the results just obtained, we find, since (fig. 9) $d\theta\sin i'\sin(\psi'-\sigma') = d\theta\sin i\sin(\psi-\sigma)$,

$$\frac{dJ}{dt} = h\mathfrak{Z}r\cos(v-\psi) - \frac{1}{\cos i'}\left(\frac{dp'}{dt}\sin\psi' + \frac{dq'}{dt}\cos\psi'\right),$$
$$\frac{dN}{dt} = -n_0\alpha - \tfrac{1}{2}h\mathfrak{Z}r\cot\tfrac{1}{2}J\sin(v-\psi) + \tfrac{1}{2}\frac{\cot\frac{1}{2}J}{\cos i'}\left(\frac{dp'}{dt}\cos\psi' - \frac{dq'}{dt}\sin\psi'\right),$$
$$\frac{dK}{dt} = n_0\eta + \tfrac{1}{2}h\mathfrak{Z}r\tan\tfrac{1}{2}J\sin(v-\psi) + \tfrac{1}{2}\frac{\tan\frac{1}{2}J}{\cos i'}\left(\frac{dp'}{dt}\cos\psi' - \frac{dq'}{dt}\sin\psi'\right) \quad (25).$$

219. The final transformation is made by changing from the variables J, N to P, Q, where

$$P = 2\sin\tfrac{1}{2}J\sin(N - N_0), \qquad Q = 2\sin\tfrac{1}{2}J\cos(N - N_0) \ldots\ldots (26).$$

In these, N_0 denotes the constant part of N. We deduce

$$\begin{aligned} dP &= dJ\cos\tfrac{1}{2}J\sin(N - N_0) + 2dN\sin\tfrac{1}{2}J\cos(N - N_0) \\ &= dJ\cos\tfrac{1}{2}J\sin(N - N_0) + QdN, \end{aligned}$$

$$\begin{aligned} dQ &= dJ\cos\tfrac{1}{2}J\cos(N - N_0) - 2dN\sin\tfrac{1}{2}J\sin(N - N_0) \\ &= dJ\cos\tfrac{1}{2}J\sin(N - N_0) - PdN. \end{aligned}$$

Substituting for dJ, dN their values just obtained, we find

$$\left.\begin{aligned} \frac{dP}{dt} &= -n_0\alpha Q - h\mathfrak{Z}r\cos\tfrac{1}{2}J\sin(v-\psi-N+N_0) + \frac{\cos\frac{1}{2}J}{\cos i'}\left(\frac{dp'}{dt}\cos\mu' + \frac{dq'}{dt}\sin\mu'\right), \\ \frac{dQ}{dt} &= n_0\alpha P + h\mathfrak{Z}r\cos\tfrac{1}{2}J\cos(v-\psi-N+N_0) + \frac{\cos\frac{1}{2}J}{\cos i'}\left(\frac{dp'}{dt}\sin\mu' - \frac{dq'}{dt}\cos\mu'\right) \end{aligned}\right\} \ldots\ldots\ldots (27),$$

where $$\mu' = -\psi' - N + N_0.$$

The equations (25), (27) for K, P, Q, are those required. The angle ψ' may be eliminated from (25) by means of the equation

$$2\sin\tfrac{1}{2}J\left(\frac{dp'}{dt}\cos\psi' - \frac{dq'}{dt}\sin\psi'\right) = \left(Q\frac{dp'}{dt} + P\frac{dq'}{dt}\right)\cos\mu' + \left(Q\frac{dq'}{dt} - P\frac{dp'}{dt}\right)\sin\mu' \ldots\ldots\ldots (27'),$$

which follows immediately from the definitions of P, Q, μ'.

When all the disturbing forces are omitted, we have K, P, Q constant and therefore N, J constant, for α, η are of the order of the disturbing forces. Now, by definition, N_0 is the constant part of N; let J_0 be the constant part of J and K_0 that of K. Hence:

The first approximation to P, Q, K *is given by*

$$P = 0, \quad Q = 2\sin\tfrac{1}{2}J_0, \quad K = K_0 \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

These values correspond to fixed positions of the orbits of the Sun and the Moon.

The quantities p, q, defined in Art. 217, have not been used. It is evident from the definitions that the equations for P, Q, K should be symmetrical (except with regard to signs) with respect to the quantities referring to the Sun and the Moon; the parts of these equations dependent on $\mathfrak{Z}$ can, in fact, be exhibited in terms of dp/dt, dq/dt by expressions similar to those which contain dp'/dt, dq'/dt. The reason for not expressing them in this form is that the latter are known functions while the former are functions of the quantities we wish to find. It will be seen from fig. 9 that p, q are the sines of the latitudes of the points X, Y below the plane of reference and that p', q' are those of X', Y' below the same plane.

Since J is small and since $N-N_0$ contains only periodic terms dependent on the disturbing forces, equations (26) show that the principal part played by Q is to bear the periodic variations in the inclination of the Moon's orbit to the ecliptic. The quantity P is small and it carries chiefly the perturbations included in N; they are multiplied by the small quantity $\sin \frac{1}{2}J$. Also, by equations (21), N_0-N+K_0-K, N_0-N-K_0+K contain the periodic parts of the motions of the Moon's node, along the Moon's orbit and along the ecliptic, respectively; the difference between these is very small.

The Form of the Development of the Disturbing Function.

220. We have from Art. 107, after replacing R by μR,

$$R = \frac{m'}{\mu}\frac{r^2}{r'^3}(\tfrac{3}{2}S^2 - 1) + \frac{m'}{\mu}\frac{r^3}{r'^4}(\tfrac{5}{2}S^3 - \tfrac{3}{2}S) + \ldots.$$

As before, S is the cosine of the angular distance between the Sun and Moon and therefore, by fig. 9,

$$S = \cos \Omega M \cos \Omega m' + \sin \Omega M \sin \Omega m' \cos J.$$

In order to obtain a perfectly general development of R, the *auxiliary* (not the instantaneous) ellipse, with its variable mean anomaly $n_0 z$, is used for the developments in the plane of the orbit, and the instantaneous values of i, θ, σ (or of the variables replacing these) for those of the plane of the orbit. For symmetry, we suppose the Sun's motion to be defined also by an auxiliary ellipse with a mean anomaly $n_0' z'$, the perturbations of its radius vector being denoted by ν' and the mean motion of its auxiliary perigee in the plane of the orbit by $n_0 y'$.

To develope R we have

$$r = \bar{r}\,(1+\nu), \quad \Omega M = v - \psi = \bar{f} + n_0 y t + \pi_0 - \psi,$$

$$r' = \bar{r}'\,(1+\nu'), \quad \Omega m' = v' - \psi' = \bar{f}' + n_0 y' t + \pi_0' - \psi',$$

by Art. 209. Substituting, we obtain

$$R = \frac{m'}{\mu}\frac{(1+\nu)^2}{(1+\nu')^3}\frac{\bar{r}^2}{\bar{r}'^3}(\tfrac{3}{2}S^2 - \tfrac{1}{2}) + \frac{m'}{\mu}\frac{(1+\nu)^3}{(1+\nu')^4}\frac{\bar{r}^3}{\bar{r}'^4}(\tfrac{5}{2}S^3 - \tfrac{3}{2}S) + \ldots,$$

where

$$S = \cos(\bar{f} + n_0 y t + \pi_0 - \psi)\cos(\bar{f}' + n_0 y' t + \pi_0' - \psi') + \sin(\bar{f} + n_0 y t + \pi_0 - \psi)\sin(\bar{f}' + n_0 y' t + \pi_0' - \psi')\cos J.$$

Since the variables $\bar{f}'$, $\bar{r}'$, ν', referring to the Sun's orbit, are supposed to be *known* functions of the time, R is thus expressed as a function of t and of the *unknown* variables $\bar{f}$, $\bar{r}$, ν, ψ, ψ', J.

Let $\qquad \omega = n_0 y t + \pi_0 - \psi, \qquad \omega' = n_0 y' t + \pi_0' - \psi' \ldots\ldots\ldots\ldots\ldots(29),$

so that, by fig. 9, ω, ω' are the distances of the apses of the *auxiliary* orbits from the common node Ω. We have then

$$S = \cos(\bar{f} + \omega)\cos(\bar{f}' + \omega') + \sin(\bar{f} + \omega)\sin(\bar{f}' + \omega')\cos J \ldots (30).$$

Now $\bar{f}$, $\bar{r}$ are, by equations (8), the elliptic true anomaly and radius vector corresponding to a mean anomaly $n_0 z$, with constants a_0, n_0, e_0; in the same way, $\bar{f}'$, $\bar{r}'$ correspond to a mean anomaly $n_0' z'$ with constants a_0', n_0', e_0'. Therefore, putting $m' a_0^3/\mu a_0'^3 = m_1^2$, we find

$$a_0 R = m_1^2 \frac{(1+\nu)^2}{(1+\nu')^3}\left(\frac{\bar{r}}{a_0}\right)^2\left(\frac{a_0'}{\bar{r}'}\right)^3\left(\tfrac{3}{2}S^2 - \tfrac{1}{2}\right) + m_1^2 \frac{(1+\nu)^3}{(1+\nu')^4}\left(\frac{\bar{r}}{a_0}\right)^3\left(\frac{a_0'}{\bar{r}'}\right)^4\left(\tfrac{5}{2}S^3 - \tfrac{3}{2}S\right) + \ldots,$$

where S has the value (30).

Comparing these values of R, S with those given in Art. 124, we see, by the remarks just made, that the method of development, given in Art. 125, will be available if we simply replace a, e, a', e', w, w' by a_0, e_0, a_0', e_0', $n_0 z$, $n_0' z'$, respectively, and multiply

$$R^{(1)} \text{ by } (1+\nu)^2/(1+\nu')^3, \quad R^{(2)} \text{ by } (1+\nu)^3/(1+\nu')^4, \text{ etc.}$$

Finally, to take into account the correction noted in Art. 7, we must further multiply $R^{(2)}$ by $(E-M)/(E+M)$.

If we look at the developments of $1/r'$, $1/\Delta$ given in Art. 5, it is not difficult to see that the general form of the multiplier of $R^{(j)}$, necessary when the force function given in Art. 8 is used, is

$$(E+M)\left\{\frac{1}{E}\left(\frac{E}{E+M}\right)^{j+1} + \frac{1}{M}\left(\frac{-M}{E+M}\right)^{j+1}\right\} = \left(\frac{E}{E+M}\right)^j - \left(\frac{-M}{E+M}\right)^j.$$

This expression was first obtained in an indirect manner by Harzer*.

221. We shall thus have the development of R in a perfectly general form: it will be expressed as an explicit function of the *unknown* variables z, ν, ω, ω', J, and of t through the *known* variables z', ν'—the rest of the symbols present being absolute constants. It will be shown later how $\mathfrak{P}$, $\mathfrak{T}$ are obtained from this development of R. In order to find R in a form suitable for the determination of the motion of the plane of the orbit, we must transform from the variables ω, ω', J to P, Q, K.

By equation (14) of Art. 124, we see that ω and ω' will only occur in R in the form of multiples of $\omega \pm \omega'$, and that a term containing in its argument $j_1(\omega + \omega')$, where j_1 is a positive integer, will have its coefficient at least of the order $\sin^{2j_1}\tfrac{1}{2}J$. Hence, all terms in R are of the form

$$m_1^2 A \sin^{2j_1}\tfrac{1}{2}J \cos\{j n_0 z + j' n_0' z' + j_1(\omega + \omega') + j_1'(\omega - \omega')\},$$

where A contains only integral powers of e_0, e_0', a_0/a_0', $\sin^2\tfrac{1}{2}J$, and j, j', j_1' are positive or negative integers.

* *Ueber die Rückwerkung der von dem Monde in der Bewegung der Sonne erzeugten Störungen auf die Bewegung des Mondes*, Astr. Nach. Vol. CXXIII. Cols, 193-200,

But, from (29) and (20), we deduce

$$\omega + \omega' = n_0 t (y + y' + 2\alpha) + 2N, \qquad \omega - \omega' = n_0 t (y - y' - 2\eta) + 2K;$$

and therefore the general term is of the form

$$m_1{}^2 A \sin^{2j_1} \tfrac{1}{2} J \cos \{\beta + 2j_1 (N - N_0)\},$$

where

$$\beta = j n_0 z + j' n_0' z' + j_1 n_0 t (y + y' + 2a) + 2j_1 N_0 + j_1' n_0 t (y - y' - 2\eta) + 2j_1' K.$$

Also, from (26), we have

$$4 \sin^2 \tfrac{1}{2} J = P^2 + Q^2, \qquad 2 \sin^2 \tfrac{1}{2} J \sin 2 (N - N_0) = PQ,$$

$$4 \sin^2 \tfrac{1}{2} J \cos 2 (N - N_0) = Q^2 - P^2.$$

The expression of R as a sum of periodic terms therefore contains the five unknowns $n_0 z$, ν, P, Q, K; of these, the variables $n_0 z$, K occur in the arguments only and the variables ν, P, Q in the coefficients only.

It is to be noticed that, since $n_0 z$, ν enter into R only through v, r, we can express R as a function of the time and of the *five* variables r, v, P, Q, K. We then have $\mathfrak{P} = \partial R / \partial r$, $\mathfrak{T} r = \partial R / \partial v$, and the expressions for the disturbing forces may therefore be directly inserted into equation (18). But as the latter has to be solved by continued approximation, this process would necessitate the expansion of the *two* expressions $\partial R / \partial r$, $\partial R / \partial v$. In the first approximation, the latter can be transformed into the differentials of R with respect to certain quantities present explicitly in the expansion of R. We shall first find relations between $\mathfrak{Z}$ and the partial derivatives of R with respect to P, Q, K, since the results for these are quite general.

222. The general expression for R is

$$R = \frac{m'}{\mu} \left(\frac{1}{\Delta} - \frac{XX' + YY' + ZZ'}{r'^3} \right) = \frac{m'}{\mu} \left(\frac{1}{\Delta} - \frac{rS}{r'^2} \right),$$

where $\qquad \Delta^2 = (X - X')^2 + (Y - Y')^2 + (Z - Z')^2 = r^2 + r'^2 - 2rr'S;$

(X, Y, Z), (X', Y', Z') being the coordinates referred to any axes.

From the first form of expression we deduce

$$\mathfrak{Z} = \frac{\partial R}{\partial Z} = \frac{m'}{\mu} \left(\frac{Z' - Z}{\Delta^3} - \frac{Z'}{r'^3} \right) = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) Z',$$

when we take the axis of Z perpendicular to the plane of the Moon's instantaneous orbit. And, from the second form of expression for R,

$$\frac{\partial R}{\partial S} = \frac{m'}{\mu} \left(\frac{rr'}{\Delta^3} - \frac{r}{r'^2} \right) = \frac{m'}{\mu} \left(\frac{1}{\Delta^3} - \frac{1}{r'^3} \right) rr'.$$

Hence
$$\mathfrak{Z} = \frac{Z'}{rr'} \frac{\partial R}{\partial S},$$

or, since (fig. 9) $Z' = -r' \sin(v' - \psi') \sin J$, the relation becomes

$$\mathfrak{Z} r = -\frac{\partial R}{\partial S} \sin(v' - \psi') \sin J \quad\quad (31).$$

223. Again, from Art. 220,

$$\begin{aligned} S &= \cos(v - \psi)\cos(v' - \psi') + \sin(v - \psi)\sin(v' - \psi')\cos J \\ &= \cos^2 \tfrac{1}{2}J \cos(v - v' - \psi + \psi') + \sin^2 \tfrac{1}{2}J \cos(v + v' - \psi - \psi'); \end{aligned}$$

and, by equations (20),

$$-\psi + \psi' = 2K - \pi_0 + \pi_0' - 2n_0\eta t, \qquad -\psi - \psi' = 2N - \pi_0 - \pi_0' + 2n_0 \alpha t.$$

Therefore, as N, K enter into S only through ψ, ψ',

$$\frac{\partial S}{\partial K} = -2\cos^2 \tfrac{1}{2}J \sin(v - v' - \psi + \psi'), \quad \frac{\partial S}{\partial N} = -2\sin^2 \tfrac{1}{2}J \sin(v + v' - \psi - \psi').$$

Again, since J only occurs in S in its explicit form,

$$\frac{\partial S}{\partial J} = -\sin J \sin(v - \psi)\sin(v' - \psi').$$

But we easily deduce from (26) by differentiation,

$$\frac{\partial S}{\partial J} = \tfrac{1}{2}\cot \tfrac{1}{2}J\left(P\frac{\partial S}{\partial P} + Q\frac{\partial S}{\partial Q}\right), \qquad \frac{\partial S}{\partial N} = Q\frac{\partial S}{\partial P} - P\frac{\partial S}{\partial Q} \quad \ldots\ldots (32).$$

Equating the two values of $\partial S/\partial J$, we obtain

$$P\frac{\partial S}{\partial P} + Q\frac{\partial S}{\partial Q} = -4\sin^2 \tfrac{1}{2}J \sin(v - \psi)\sin(v' - \psi') \quad \ldots\ldots (33).$$

Also, from the values of $\partial S/\partial K$, $\partial S/\partial N$, we find

$$\frac{\partial S}{\partial N}\cos^2 \tfrac{1}{2}J - \frac{\partial S}{\partial K}\sin^2 \tfrac{1}{2}J = -\sin^2 J \cos(v - \psi)\sin(v' - \psi').$$

Multiply this equation by Q, the first equation for $\partial S/\partial J$ by $P \sin J$ and add. We obtain, after using the values (26) of P and Q in the right hand members,

$$\begin{aligned} P\frac{\partial S}{\partial J}\sin J + Q\frac{\partial S}{\partial N}\cos^2 \tfrac{1}{2}J - Q\frac{\partial S}{\partial K}\sin^2 \tfrac{1}{2}J& \\ = -2\sin^2 J \sin \tfrac{1}{2}J \cos(v - \psi - N + N_0)&\sin(v' - \psi'). \end{aligned}$$

But by (32),

$$P\frac{\partial S}{\partial J}\sin J + Q\frac{\partial S}{\partial N}\cos^2 \tfrac{1}{2}J = (P^2 + Q^2)\frac{\partial S}{\partial P}\cos^2 \tfrac{1}{2}J;$$

and therefore we have, after dividing by $P^2+Q^2=4\sin^2\frac{1}{2}J$,

$$\frac{\partial S}{\partial P}\cos^2\tfrac{1}{2}J-\tfrac{1}{4}Q\frac{\partial S}{\partial K}=-\sin J\cos\tfrac{1}{2}J\cos(v-\psi-N+N_0)\sin(v'-\psi')\ldots(34).$$

In a similar manner we can deduce

$$\frac{\partial S}{\partial Q}\cos^2\tfrac{1}{2}J+\tfrac{1}{4}P\frac{\partial S}{\partial K}=-\sin J\cos\tfrac{1}{2}J\sin(v-\psi-N+N_0)\sin(v'-\psi')\ldots(35).$$

Since P, Q, K enter into R only through S, we have

$$\frac{\partial R}{\partial P}=\frac{\partial R}{\partial S}\frac{\partial S}{\partial P},\quad \frac{\partial R}{\partial Q}=\frac{\partial R}{\partial S}\frac{\partial S}{\partial Q},\quad \frac{\partial R}{\partial K}=\frac{\partial R}{\partial S}\frac{\partial S}{\partial K}.$$

Whence, multiplying the equations (33), (34), (35) by $\partial R/\partial S$ and substituting in their right hand members the value of $\partial R/\partial S$ given by (31), we find

$$\left.\begin{aligned}P\frac{\partial R}{\partial P}+Q\frac{\partial R}{\partial Q}&=2r\mathfrak{Z}\tan\tfrac{1}{2}J\sin(v-\psi),\\ \frac{\partial R}{\partial P}\cos^2\tfrac{1}{2}J-\tfrac{1}{4}Q\frac{\partial R}{\partial K}&=r\mathfrak{Z}\cos\tfrac{1}{2}J\cos(v-\psi-N+N_0),\\ \frac{\partial R}{\partial Q}\cos^2\tfrac{1}{2}J+\tfrac{1}{4}P\frac{\partial R}{\partial K}&=r\mathfrak{Z}\cos\tfrac{1}{2}J\sin(v-\psi-N+N_0)\end{aligned}\right\}\ldots\ldots\ldots(36).$$

These results, which are quite general, are put into the form in which they will be useful in Art. 235.

The First Approximation to R and to the disturbing forces.

224. A limitation of the general value of R will now be made by supposing the orbit of the Sun in its instantaneous plane to be an ellipse, so that $n_0'z'=g'=n_0't+c_0'$ (where c_0' is a constant) and $\nu'=0$, $\bar{r}'=r'$. The perturbations of the solar orbit, thus neglected, only produce small effects on the motion of the Moon.

The first approximation to R is obtained by substituting for the coordinates their elliptic values, that is, we put

$$n_0z=g=n_0t+c_0,\quad \nu=0,\quad K=K_0,\quad N=N_0,\quad J=J_0;$$

whence

$$\omega+\omega'=n_0t(y+y'+2\alpha)+2N_0,$$
$$\omega-\omega'=n_0t(y-y'-2\eta)+2K_0,$$

in which y', being a known constant, is retained: y, α, η are constants to be found. We have also, by (26),

$$P=0,\quad Q=2\sin\tfrac{1}{2}J_0.$$

The first approximation to R is therefore expressed explicitly as a function of the time, the arguments being sums of multiples of the four angles g, g', ω, ω', and the coefficients being expanded in powers of e_0, e_0',

$\sin^2 \frac{1}{2} J_0$, a_0/a_0'. For the motion of the plane of the orbit when R is expressed in terms of $n_0 z$, ν, P, Q, K, we also put $n_0 z = g$, $\nu = 0$. The partial derivatives of R with respect to P, Q must be formed before we give to P the value 0 and to Q the value $2 \sin \frac{1}{2} J_0$. As regards K, we evidently have $\partial R/\partial K = \partial R/\partial K_0$.

In the *Fundamenta* (p. 81), Hansen used the derivatives of R with regard to p, q as though R were a function of the *four* variables r, v, p, q only. The difficulty is merely that Hansen has attached a meaning to $\partial R/\partial p$, $\partial R/\partial q$ which is unusual. The point was cleared up by Jacobi*.

225. *To express the disturbing forces $\mathfrak{P}_0$, $\mathfrak{T}_0$ in terms of the partial derivatives of R.*

Denote by $\mathfrak{P}_0$, $\mathfrak{T}_0$ the values of the disturbing forces $\mathfrak{P}$, $\mathfrak{T}$ when the first approximation to R is used. In the terms multiplied by quantities of the order of the disturbing forces, we can put $r = \bar{r} = r_0$, $f = \bar{f} = f_0$, where r_0, f_0 are the values of $\bar{r}, \bar{f}$ when $n_0 z = g$. Since, in the first approximation, e_0, g enter into R only through $\bar{r}, \bar{f}$ and since g enters here in the same way that $nt + \epsilon$ entered in the expressions of Art. 75, the second and third of equations (4) of that article are available. We therefore obtain, with the necessary changes in notation,

$$\left.\begin{aligned} \frac{\partial R}{\partial e_0} &= -\mathfrak{P}_0 a_0 \cos f_0 + \mathfrak{T}_0 r_0 \left(\frac{a_0}{r_0} \sin f_0 + \frac{\sin f_0}{1 - e_0^2}\right), \\ \frac{\partial R}{\partial g} = \frac{\partial R}{\partial \epsilon} &= \mathfrak{P}_0 \frac{a_0 e_0}{\sqrt{1 - e_0^2}} \sin f_0 + \mathfrak{T}_0 \frac{a_0^2}{r_0} \sqrt{1 - e_0^2} \\ &= \mathfrak{P}_0 \frac{a_0 e_0}{\sqrt{1 - e_0^2}} \sin f_0 + \mathfrak{T}_0 r_0 \left[\frac{a_0 e_0 \cos f_0}{r_0 \sqrt{1 - e_0^2}} + \frac{1 + e_0 \cos f_0}{(1 - e_0^2)^{\frac{3}{2}}}\right] \end{aligned}\right\} \ldots (37).$$

By means of these results we can express $\mathfrak{P}_0$, $\mathfrak{T}_0$ in terms of $\partial R/\partial e_0$, $\partial R/\partial g$.

Also, since ω enters into R only in the form $v + \omega$, we have

$$\mathfrak{T}_0 r_0 = \frac{\partial R}{\partial v} = \frac{\partial R}{\partial \omega} \quad \ldots\ldots (38).$$

The First Approximation to W.

226. The general process of solution adopted by Hansen is one of continued approximation. There has been found, in Art. 215, a general expression for dW/dt which contains $\mathfrak{P}$, $\mathfrak{T}$. Now W is of the order of the disturbing forces and all the terms present in the expression for dW/dt are implicitly functions of z, ζ, t.

* *Auszug zweier Schreiben etc.* Crelle, Vol. XLII. pp. 12–31.

Put

$$n_0 z = n_0 t + c_0 + \delta z = g + n_0 \delta z, \qquad n_0 \zeta = n_0 \tau + c_0 + \delta \zeta = \gamma + n_0 \delta \zeta \ldots (39),$$

and, in finding the first approximation to W, neglect δz, $\delta \zeta$. Let ρ_0, ϕ_0, f_0, W_0, etc. be the values of $\bar{\rho}$, $\bar{\phi}$, $\bar{f}$, W, etc. when for $n_0 z$, $n_0 \zeta$ are put g, γ, respectively. Also, as h, r differ from h_0, r_0 by quantities of the order of the disturbing forces, we can, in the terms multiplied by quantities of that order, put $h = h_0$, $r = r_0$.

Let

$$T_0 = \frac{h_0}{n_0} \mathfrak{T}_0 r_0 \left[2\frac{\rho_0}{r_0} \cos(f_0 - \phi_0) - 1 + \frac{2\rho_0}{a_0(1-e_0^2)} \{\cos(f_0 - \phi_0) - 1\} \right] + 2\frac{h_0}{n_0} \rho_0 \mathfrak{P}_0 \sin(f_0 - \phi_0).$$

The equation for W_0 may be written

$$\frac{dW_0}{dt} = n_0 T_0 + \frac{n_0 y}{\sqrt{1-e_0^2}} \left[\left(\frac{\rho_0}{a_0}\right)^2 \frac{\partial W_0}{\partial \gamma} - \tfrac{1}{2} \left(W_0 + \frac{h_0}{h} + 1 \right) \frac{d}{d\gamma} \left(\frac{\rho_0}{a_0}\right)^2 \right].$$

But since y is of the order of the disturbing forces, we can, in the coefficient of y, neglect W_0 and put $h = h_0$. Hence, the first approximation will be obtainable from

$$\frac{dW_0}{dt} = n_0 T_0 - \frac{n_0 y}{\sqrt{1-e_0^2}} \frac{d}{d\gamma} \left(\frac{\rho_0}{a_0}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

227. We shall now transform T_0 so that the values of $\mathfrak{P}_0$, $\mathfrak{T}_0$, given by (37), (38), may be inserted. The suffix zero, which occurs in *every* symbol present in T_0, will, for the sake of brevity, be omitted until the end of this article.

With this understanding we have, by the elliptic formulæ (16),

$$-1 - \frac{2\rho}{a(1-e^2)} = -3 + 2\frac{l - \rho}{a(1-e^2)} = -3 + \frac{2\rho e \cos\phi}{a(1-e^2)}, \qquad h = \frac{na}{\sqrt{1-e^2}}.$$

Therefore

$$T = -\frac{3}{\sqrt{1-e^2}} a\mathfrak{T} r + \frac{2\rho\cos\phi}{\sqrt{1-e^2}} \left[\left(\frac{a\cos f}{r} + \frac{\cos f + e}{1-e^2} \right) \mathfrak{T} r + a\mathfrak{P}\sin f \right]$$

$$+ \frac{2\rho\sin\phi}{\sqrt{1-e^2}} \left[\left(\frac{a\sin f}{r} + \frac{\sin f}{1-e^2} \right) \mathfrak{T} r - a\mathfrak{P}\cos f \right]$$

$$= -\frac{3}{\sqrt{1-e^2}} \left[\left(\frac{ae\cos f}{r} + \frac{1 + e\cos f}{1-e^2} \right) a\mathfrak{T} r + a^2\mathfrak{P} e\sin f \right]$$

$$+ \frac{2\rho\cos\phi + 3ae}{a\sqrt{1-e^2}} \left[\left(\frac{a\cos f}{r} + \frac{\cos f + e}{1-e^2} \right) a\mathfrak{T} r + a^2\mathfrak{P}\sin f \right]$$

$$+ \frac{2\rho\sin\phi}{a\sqrt{1-e^2}} \left[\left(\frac{a\sin f}{r} + \frac{\sin f}{1-e^2} \right) a\mathfrak{T} r - a^2\mathfrak{P}\cos f \right].$$

We deduce, from (37),

$$\frac{\partial aR}{\partial g} - \frac{a\mathfrak{T}r}{\sqrt{1-e^2}} = \frac{a^2 e}{\sqrt{1-e^2}} \mathfrak{P} \sin f + \frac{ae}{\sqrt{1-e^2}} \left[\frac{a\cos f}{r} + \frac{\cos f + e}{1-e^2}\right] \mathfrak{T}r.$$

On the left-hand side of this equation put $\mathfrak{T}r = \partial R/\partial\omega$ and substitute in the second line of the latter expression for T: for the first and third lines, the equations (37) are available. We obtain, on restoring the suffixes,

$$T_0 = -3\frac{\partial a_0 R}{\partial g} + \frac{1}{e_0}\left(\frac{2\rho_0 \cos\phi_0}{a_0} + 3e_0\right)\left(\frac{\partial a_0 R}{\partial g} - \frac{1}{\sqrt{1-e_0^2}}\frac{\partial a_0 R}{\partial \omega}\right) + \frac{2\rho_0 \sin\phi_0}{a_0\sqrt{1-e_0^2}}\frac{\partial a_0 R}{\partial e_0}.$$

This expression is now very easily calculated from the first approximation to the value of $a_0 R$, for R has been expressed as a sum of periodic terms whose arguments contain g, ω and whose coefficients are functions of e_0. The portions dependent on R are thus expressed by means of periodic series with constant coefficients and with arguments of the form $\beta t + \beta'$.

228. Let, for a moment,

$$T_0 = F' - \left(\frac{\rho_0 \cos\phi_0}{a_0} + \tfrac{3}{2}e_0\right) G' - \frac{e_0\rho_0 \sin\phi_0}{a_0\sqrt{1-e_0^2}} H',$$

in which the signification of F', G', H' is evident. Since $a_0 R$ is expressible by means of cosines, and since g, ω occur in the arguments only while e_0 occurs in the coefficients only, F', G' will be developable in sines and H' in cosines of angles which are all of the form $\beta t + \beta'$ (β, β' constant). Here, $\beta t + \beta'$ is formed of multiples of the angles g, g', ω, ω', all of which, owing to the introduction of α, η, contain t; also, $\beta' = 0$ when $\beta = 0$, for n_0, n_0', etc. are supposed to be incommensurable with one another.

Since ρ_0, ϕ_0 are the radius vector and true anomaly corresponding to a mean anomaly γ, we have, by the theorem of Art. 43,

$$T_0 = \Sigma \sum_{-\infty}^{\infty} C_j \sin(j\gamma + \beta t + \beta').$$

The first sign of summation refers to the angles $\beta t + \beta'$ and the second to the integral values of j; C is the symbol for the general coefficient corresponding to the angle $\beta t + \beta'$. The extra labour, caused by the presence of the angle γ (which does not occur in $\beta t + \beta'$), is compensated by the ease with which the other coefficients can be obtained, when the values of C_0, C_1, C_{-1}, for all values of $\beta t + \beta'$, have been calculated. See Art. 43.

When T_0 has been thus found in terms of the time, we obtain from (40)

$$\frac{dW_0}{n_0 dt} = \Sigma \sum_{-\infty}^{\infty} C_j \sin(j\gamma + \beta t + \beta') - \frac{y}{\sqrt{1-e_0^2}}\frac{d}{d\gamma}\left(\frac{\rho_0}{a_0}\right)^2.$$

But, by equation (19) of Art. 43, we have

$$\frac{d}{d\gamma}\left(\frac{\rho_0}{a_0}\right)^2 = -2\sum_1^\infty jR_j \sin j\gamma.$$

Hence

$$\frac{dW_0}{n_0 dt} = \Sigma \sum_{-\infty}^{\infty} c_j \sin(j\gamma + \beta t + \beta') + \sum_1^\infty \left[\frac{2y}{\sqrt{1-e_0^2}} jR_j + c_j' - c'_{-j}\right] \sin j\gamma,$$

where the terms for which $\beta t + \beta' = 0$ are written separately, their coefficients being denoted by c_j'.

229. *Integration of the Equation for W_0 and Determination of y.*

We have on integration, since γ is constant during the process,

$$W_0 = -n_0 \Sigma \sum_{-\infty}^{\infty} \frac{c_j}{\beta} \cos(j\gamma + \beta t + \beta') + n_0 t \sum_1^\infty \left[\frac{2y}{\sqrt{1-e_0^2}} jR_j + c_j' - c'_{-j}\right] \sin j\gamma + F(\gamma),$$

where the additive arbitrary constant, denoted by $F(\gamma)$, may be a function of γ.

Putting $\tau = t$ and therefore $\gamma = g$, we find

$$\overline{W}_0 = -n_0 \Sigma \sum_{-\infty}^{\infty} \frac{c_j}{\beta} \cos(jg + \beta t + \beta') + n_0 t \sum_1^\infty \left[\frac{2y}{\sqrt{1-e_0^2}} jR_j + c_j' - c'_{-j}\right] \sin jg + F(g) \quad \text{.........(41).}$$

Now y was specially introduced in order that expressions of the form $t \times$ *periodic term* might be eliminated, if possible. Determine y so that the coefficient of $t \sin g$ vanishes. This gives

$$2yR_1/\sqrt{1-e_0^2} + c_1' - c'_{-1} = 0 \quad \text{.....................(42).}$$

The corollary to the theorem of Art. 43, applied to $c_j' - c'_{-j}$, then shows that

$$2yjR_j/\sqrt{1-e_0^2} + c_j' - c'_{-j} = 0.$$

Hence the coefficient of $t \sin jg$ vanishes, and therefore all the terms having the time as a factor disappear from $\overline{W}_0$.

The equation (42) gives a first approximation to y*. That it should be capable of being determined so that *all* terms of the forbidden form may be eliminated, is sufficiently evident from what has been said in previous chapters. All that now remains is the determination of the form of $F(g)$, the function which contains the arbitrary constants.

* *Fundamenta*, p. 191. The determination of y, given in the *Darlegung*, I. p. 338, is not directly applicable here because Hansen is there simply performing a verification of his tables.

230. *Determination of the Form of $F(g)$.*

The constants are found by considering the initial form of W. Put

$$\bar{\rho} = a_0(1 - e_0^2) - \bar{\rho} e_0 \cos \bar{\phi},$$

and express W_0 as a linear function of $\bar{\rho} \cos \bar{\phi}$, $\bar{\rho} \sin \bar{\phi}$; the coefficients of these quantities being of the order of the disturbing forces, we can put ρ_0, ϕ_0 for $\bar{\rho}$, $\bar{\phi}$. Hence, the expression (17) may be written

$$\left.\begin{aligned} W_0 &= \Xi + \Upsilon\left(\frac{\rho_0}{a_0}\cos\phi_0 + \tfrac{3}{2}e_0\right) + \Psi\frac{\rho_0}{a_0}\sin\phi_0, \\ \text{where} \quad \Xi &= -1 - \frac{h_0}{h} + 2\frac{h}{h_0} - 3e_0\frac{h}{h_0}\frac{e\cos(\chi - n_0 y t - \pi_0) - e_0}{1 - e_0^2}, \\ \Upsilon = 2\frac{h}{h_0}\frac{e\cos(\chi - n_0 y t - \pi_0) - e_0}{1 - e_0^2}, &\qquad \Psi = 2\frac{h}{h_0}\frac{e\sin(\chi - n_0 y t - \pi_0)}{1 - e_0^2} \end{aligned}\right\} \ldots(43).$$

Since the arbitrary $F(\gamma)$ is the only constant in the expression for W_0 given in the previous article, all that is required is to find the form of the constant part of W_0. Let the constant part of χ (which, by Art. 206, is the distance $X\pi$) be π_0; then Ψ will contain no constant term.

The constant parts of h, e are as yet undefined: it was merely assumed (Art. 216) that they differed from h_0, e_0 by quantities of the order of the disturbing forces. In Art. 226 these differences were multiplied by quantities of the order of the disturbing forces and they were therefore neglected; here this does not take place. Let the differences be such that the constant parts of Ξ, Υ are denoted by b, ξ; the approximate constant part of $e - e_0$ is then $\frac{1}{2}\xi$ and that of $h/h_0 - 1$ is $\frac{1}{3}b + \frac{1}{2}e_0\xi$. (See Art. 234.)

The constant part of W_0 is therefore

$$F(\gamma) = b + \xi\left(\frac{\rho_0}{a_0}\cos\phi_0 + \tfrac{3}{2}e_0\right) = b - \xi\sum_1^\infty \frac{\partial R_j}{\partial e_0}\cos j\gamma,$$

by equation (18), Art. 43. From this we deduce $F(g)$ by putting $\tau = t$ or $\gamma = g$.

Since the terms multiplied by t have been made to vanish, the equation (41) becomes, on the substitution of this value of $F(g)$,

$$\overline{W}_0 = -n_0\Sigma\sum_{-\infty}^{\infty}\frac{C_j}{\beta}\cos(jg + \beta t + \beta') + b - \xi\sum_1^\infty\frac{\partial R_j}{\partial e_0}\cos jg.$$

On the subject of the determination of the constants b, ξ and on their general meaning with respect to h, e and h_0, e_0, Hansen's paper, referred to in the foot-note of page 168 above, will be found of great assistance. See also *Fundamenta*, pp. 65, 66, 196, *Darlegung*, I. pp. 332—337, etc.

The Integration of the Equations for z, v, and the Signification to be attached to the Constants of the Auxiliary Ellipse.

231. Substituting the first approximation $\overline{W}_0$ for $\overline{W}$ in (12) and neglecting the third term of the equation—that term being of the order of the square of the disturbing forces,—we find

$$\frac{dz}{dt} = 1 - n_0 \Sigma \sum_{-\infty}^{\infty} \frac{C_j}{\beta} \cos(jg + \beta t + \beta') + b - \xi \sum_{1}^{\infty} \frac{\partial R_j}{\partial e_0} \cos jg$$
$$- y(1 - e_0^2)^{-\frac{1}{2}}(1 + \tfrac{3}{2}e_0^2 + 2\sum_{1}^{\infty} R_j \cos jg) \ldots\ldots\ldots (44),$$

where, in the last term, r_0 has been put for $\bar{r}$ (since y is of the order of the disturbing forces) and for r_0 has been inserted its value given by equation (19) of Art. 43. We can also put for y the value found in Art. 229.

The non-periodic part of dz/dt will be

$$1 + b + d,$$

where b, d are of the order of the disturbing forces, b being arbitrary and d containing the rest of the known non-periodic terms in (44). On integration, it will produce in $n_0 z$ the term

$$n_0 t (1 + b + d).$$

There is now an opportunity of defining n_0. Let it represent *the mean motion of the mean anomaly $n_0 z$.* With this definition we must determine the arbitrary b, so that

$$b + d = 0.$$

Again, if d' represent the known coefficients of $\cos g$ in (44), the coefficient of this term will be

$$d' - \xi \frac{\partial R_1}{\partial e_0},$$

where d', ξ are of the order of the disturbing forces. On integration the coefficient of $\sin g$ in z will therefore be $(d' - \xi\, \partial R_1/\partial e_0)/n_0$. *We define ξ so that the coefficient of $\sin g$ in z vanishes.* This, as we shall see directly, amounts to a definition of e_0.

The value of z is therefore given by

$$n_0 z = g + \Sigma B \sin(\beta t + \beta'),$$

where $\beta t + \beta'$ is an angle of the form

$$jg + j'g' + j_1\omega + j_1'\omega', \qquad (j, j', j_1, j_1' = +\infty \ldots -\infty):$$

the term for which $j = 1$, $j' = j_1 = j_1' = 0$ having its coefficient zero.

The arbitrary constant present in g is the value of the non-periodic part of the disturbed mean anomaly at time $t=0$.

232. The value of n_0, defined above, is the mean motion of the mean anomaly of the auxiliary ellipse, that is, the mean rate of separation of the Moon from the perigee of the auxiliary ellipse. To obtain the mean motion of the Moon it is necessary to add to n_0 the term $n_0(y-2\eta)$ which (Art. 217) represents the mean motion of the perigee of the auxiliary ellipse. Since the mean motion of the Moon is observed directly, in order to obtain n_0 for purposes of computation we must also know the mean motion of the Moon's perigee—a quantity which is found from theory. The latter is, however, capable of being observed with great accuracy and Hansen, in performing his computations, used a value of n_0 obtained from these observed values. The computed value for the motion of the perigee agrees very nearly with the value obtained directly from observation: the small difference causes no sensible error in the coefficients of the periodic terms*.

When the value of n_0z is inserted in the expansion of $\bar{f}$ in terms of the mean anomaly, we obtain

$$\bar{f}=n_0z+e_1\sin n_0z+e_2\sin 2n_0z+\ldots,$$

where $e_1, e_2,\ldots$ are known functions of e_0 given by equation (7) of Art. 34. Since $n_0z=g+n_0\delta z$, we find

$$\bar{f}=g+n_0\delta z+\left(1-\frac{(n_0\delta z)^2}{2!}+\ldots\right)e_1\sin g+\left(\frac{n_0\delta z}{1!}-\frac{(n_0\delta z)^3}{3!}+\ldots\right)e_1\cos g+\ldots$$

Although $n_0\delta z$ contains no term with the argument g, terms with argument g, other than $e_1\sin g$, will arise in $\bar{f}$, owing to combinations of terms in the powers of $n_0\delta z$ with those of the elliptic development. For instance, the term in $n_0\delta z$ with argument $2g$ will combine with $e_1\cos g$ to produce a term of argument g in $\bar{f}$. These terms, as well as those of the same argument which arise from the reduction of the longitude in the orbit to that on the ecliptic, are very small.

Hansen computes n_0z with $e_0=\cdot05490079$; this produces $22637''\cdot15$ as the coefficient of $\sin g$ in the expression for the ecliptic longitude. The observed value of this coefficient is, according to him, $22640''\cdot15$ and, in order to produce this coefficient, the value of e_0 should have been $\cdot05490807$. The difference is very small and it is sufficiently taken into account by multiplying those terms whose characteristic is e by $\cdot05490807/\cdot05490079$. Very few terms need to be thus corrected: the principal one is the evection. (See the papers of Newcomb referred to in Art. 238 below.)

233. The equation (15) will now serve for the calculation of ν. But since $\partial\overline{W}/\partial z=\partial\overline{W}/\partial\zeta$ (where the bar denotes that τ is changed into t after the differentiation) and since $\partial W/n_0\partial\zeta=\partial W/\partial\gamma$ in the first approximation, we can put $\partial\overline{W}/\partial z=n_0\partial\overline{W_0/\partial\gamma}$. Also, to the same degree of accuracy, we can neglect the product $y\nu$ and put $d/dz=n_0d/dg$. The equation therefore becomes

$$\frac{d\nu}{dt}=-\tfrac{1}{2}n_0\overline{\frac{\partial W_0}{\partial\gamma}}+\tfrac{1}{2}\frac{n_0y}{\sqrt{1-e_0{}^2}}\frac{d}{dg}\left(\frac{r_0}{a_0}\right)^2.$$

* *Darlegung*, I. pp. 173, 348.

The value of $\partial W_0/\partial\gamma$ may be obtained from Art. 229 and thence, by putting $\tau = t$, that of the first term of this equation; therefore, after inserting the values of y, r_0^2/a_0^2 as before, we have

$$\frac{d\nu}{dt} = \Sigma A \sin(\beta t + \beta');$$

whence, integrating,

$$\nu = C - \Sigma \frac{A}{\beta} \cos(\beta t + \beta').$$

Here $\beta t + \beta'$ is of the same form as before and A is the corresponding coefficient. The constant C, owing to the relation $n_0^2 a_0^3 = \mu$, is not arbitrary: we proceed to find it.

234. We have, from equations (43), Art. 230,

$$\Xi + \tfrac{3}{2} e_0 \Upsilon = -1 - \frac{h_0}{h} + 2\frac{h}{h_0} = 2\delta\frac{h}{h_0} - \delta\frac{h_0}{h},$$

where $\delta(h/h_0)$, $\delta(h_0/h)$ denote the differences of h/h_0, h_0/h from unity. Also, in the same article, b, ξ were defined to be the constant parts of Ξ, Υ. Hence

$$b + \tfrac{3}{2} e_0 \xi = \text{constant part of} \quad 2\delta\frac{h}{h_0} - \delta\frac{h_0}{h}$$

$$= \quad \text{,,} \qquad \text{,,} \quad -3\delta\frac{h_0}{h} + 2\left(\delta\frac{h_0}{h}\right)^2 - \dots.$$

But, by definition (Art. 211),

$$\overline{W} = -1 - \frac{h_0}{h} + 2\frac{h}{h_0}\frac{\bar{r}}{a_0}\frac{1 + e\cos f}{1 - e_0^2}$$

$$= -1 - \frac{h_0}{h} + 2\frac{h}{h_0}\frac{\bar{r}}{r}\frac{l}{l_0} = -1 + \frac{h_0}{h}\frac{1-\nu}{1+\nu}$$

$$= -1 + \frac{h_0}{h}\frac{1}{(1+\nu)^2} - \frac{h_0}{h}\frac{\nu^2}{(1+\nu)^2},$$

since $h^2 l = h_0^2 l_0$ and $r = \bar{r}(1+\nu)$. Therefore

$$\delta\frac{h_0}{h} = \frac{h_0}{h} - 1 = \left\{1 + \overline{W} + \frac{h_0}{h}\frac{\nu^2}{(1+\nu)^2}\right\}(1+\nu)^2 - 1$$

$$= \overline{W} + 2\nu + \nu^2 + \frac{h_0}{h}\frac{\nu^2}{(1+\nu)^2} + (2\nu + \nu^2)\left\{\overline{W} + \frac{h_0}{h}\frac{\nu^2}{(1+\nu)^2}\right\}:$$

an equation which is true generally.

Neglecting, in the first approximation, the terms which are of the order of the square of the disturbing forces, this equation gives

$$\text{const. part of } 2\nu = \text{const. part of } \left(\delta\frac{h_0}{h} - \overline{W}_0\right)$$

$$= -\tfrac{1}{3} b - \tfrac{1}{2} e_0 \xi - \text{const. part of } \overline{W}_0.$$

As b, ξ and the constant part of $\overline{W}_0$ are already known, we can find C from this result.

Cor. When z, ν have been found, the first approximation to h_0/h can be most easily calculated by means of equation (11) which, when r has been put equal to $\bar{r}(1+\nu)$, gives

$$\frac{dz}{dt} = \frac{h_0}{h(1+\nu)^2} - \frac{y}{\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2,$$

or,

$$\frac{h_0}{h} - 1 = 2\nu + \nu^2 + (1+\nu)^2 \frac{d}{dt}\delta z + \frac{y(1+\nu)^2}{\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2.$$

This will be required in the next approximation to z.

It can also be found by integrating the equation $dh/dt = -h^2\partial R/\partial v$ *.

The values of P, Q, K as far as the First Order of the Disturbing Forces.

235. In Arts. 218, 219, the equations satisfied by P, Q, K have been given in terms of the disturbing force $\mathfrak{Z}$ and of certain known quantities p', q'. Further, in Art. 223, expressions for the partial derivatives of R with respect to P, Q, K have been found in terms of $\mathfrak{Z}$. Substitute the results (36) in the second terms of the right-hand members of equations (25), (27), and substitute the result (27′) in the third term of (25). The equations for dP/dt, dQ/dt, dK/dt become

$$\left.\begin{aligned}
\frac{dP}{dt} &= -n_0\alpha Q - h\left(\frac{\partial R}{\partial Q}\cos^2\tfrac{1}{2}J + \tfrac{1}{4}P\frac{\partial R}{\partial K}\right) + \frac{\cos\frac{1}{2}J}{\cos i'}\left(\frac{dp'}{dt}\cos\mu' + \frac{dq'}{dt}\sin\mu'\right),\\
\frac{dQ}{dt} &= n_0\alpha P + h\left(\frac{\partial R}{\partial P}\cos^2\tfrac{1}{2}J - \tfrac{1}{4}Q\frac{\partial R}{\partial K}\right) + \frac{\cos\frac{1}{2}J}{\cos i'}\left(\frac{dp'}{dt}\sin\mu' - \frac{dq'}{dt}\cos\mu'\right),\\
\frac{dK}{dt} &= n_0\eta + \tfrac{1}{4}h\left(P\frac{\partial R}{\partial P} + Q\frac{\partial R}{\partial Q}\right)\\
&\quad + \frac{1}{4\cos i'\cos\frac{1}{2}J}\left\{\left(Q\frac{dp'}{dt} + P\frac{dq'}{dt}\right)\cos\mu' + \left(Q\frac{dq'}{dt} - P\frac{dp'}{dt}\right)\sin\mu'\right\}
\end{aligned}\right\} \quad (45),$$

where $\mu' = -\psi' - N + N_0 = n_0(\alpha+\eta)t + N_0 - K - \pi_0'$, by (21).

These are the general equations for P, Q, K; they are given on p. 93 of the *Fundamenta* and on p. 117 of the first volume of the *Darlegung* †. The second of equations (21) shows that the constant part ψ_0' of ψ' is $\pi_0' - N_0 + K_0$; therefore, if we neglect the periodic part (which is very small) of K, we have $\mu' = n_0(\alpha+\eta)t - \psi_0'$. Since $-n_0(\alpha+\eta)t$ is the mean motion of the node (Art. 217), this result shows that $-\mu'$ is the mean longitude of the ascend-

* *Darlegung*, I. pp. 164, 165. See also G. W. Hill, *Note on Hansen's general Formulæ for Perturbations.* Amer. Journ. Math. Vol. IV. pp. 256–259.

† Hansen, in the *Darlegung*, denotes the angle μ' by θ. The change is made to avoid confusion with the angle $x\Omega_1$.

ing node of the Moon's orbit on the ecliptic. It is to be remembered that a, η are, by definition, to be so determined that N, K or that P, K contain no terms proportional to the time.

236. We can immediately show that the secular motion of the ecliptic will only produce periodic terms in P, Q. Let this motion be given by

$$p' = b_1 t \cos i' + \text{const.}, \qquad q' = b_1' t \cos i' + \text{const.},$$

where b_1, b_1' are constants supposed known. Substituting in (45), we see that the parts $d\delta P/dt$, $d\delta Q/dt$, due to these terms, are periodic. The corresponding terms in $d\delta K/dt$, being multiplied by P or Q and therefore by $\sin \frac{1}{2} J$, are much smaller and they may be neglected. If we put $K = K_0$ in the expression for μ', the periods of these terms will be seen to be the same as that of μ', that is, they will be $2\pi/n_0 (a + \eta)$; this quantity is the period of revolution of the Moon's node along the ecliptic* (Art. 217).

Another method of finding the effect of the motion of the ecliptic will be given in Chap. XIII.

237. It has been seen, in Art. 219, that when the disturbing forces are neglected,

$$P = 0, \qquad Q = 2 \sin \tfrac{1}{2} J_0, \qquad K = K_0.$$

Neglect the motion of the ecliptic, that is, consider p', q' as constants. Put

$$P = 0 + \delta P, \qquad Q = 2 \sin \tfrac{1}{2} J_0 + \delta Q, \qquad K = K_0 + \delta K.$$

In the terms containing the disturbing forces, neglect δP, δQ, δK and put $h = h_0$.

Let $$n_0 B_0 = -h_0 \left(\frac{\partial R}{\partial Q}\right)_0 \cos^2 \tfrac{1}{2} J_0, \qquad n_0 D_0 = \tfrac{1}{2} h_0 \left(\frac{\partial R}{\partial Q}\right)_0 \sin \tfrac{1}{2} J_0,$$

$$n_0 C_0 = h_0 \left[\left(\frac{\partial R}{\partial P}\right)_0 \cos^2 \tfrac{1}{2} J_0 - \tfrac{1}{2} \left(\frac{\partial R}{\partial K}\right)_0 \sin \tfrac{1}{2} J_0\right],$$

where the zero suffix indicates that, after R has been differentiated, the constant values of P, Q, K are to be substituted.

The equations (45) become

$$\frac{d}{dt} \delta P = -2 n_0 a \sin \tfrac{1}{2} J_0 + n_0 B_0, \qquad \frac{d}{dt} \delta Q = n_0 C_0, \qquad \frac{d}{dt} \delta K = n_0 \eta + n_0 D_0.$$

It is not difficult to see, from Art. 221, that B_0, D_0 will be expansible in series of cosines, and C_0 in series of sines of angles depending on the time; hence C_0 will contain no constant term.

Let A_0 be the constant term in $(\partial R/\partial Q)_0$: the constant term in $n_0 B_0$ will be $-h_0 A_0 \cos^2 \frac{1}{2} J_0$ and that in $n_0 D_0$ will be $\frac{1}{2} h_0 A_0 \sin \frac{1}{2} J_0$. Hence the constant terms in $d\delta P/dt$, $d\delta K/dt$ are respectively

$$-2 n_0 a \sin \tfrac{1}{2} J_0 - h_0 A_0 \cos^2 \tfrac{1}{2} J_0, \qquad n_0 \eta + \tfrac{1}{2} h_0 A_0 \sin \tfrac{1}{2} J_0.$$

* *Fundamenta*, p. 94; *Darlegung*, I. p. 118.

When the equations for δP, δK are integrated, these will be the terms multiplying the time.

Now α, η were introduced so that N, K should contain no terms proportional to the time; the condition demands that P, K contain no such terms. We therefore determine α, η so that the two expressions written above are zero. Hence

$$n_0\alpha = -\tfrac{1}{2}h_0 A_0 \cos^2\tfrac{1}{2}J_0/\sin\tfrac{1}{2}J_0, \qquad n_0\eta = -\tfrac{1}{2}h_0 A_0 \sin\tfrac{1}{2}J_0;$$

giving $$\eta = \alpha\tan^2\tfrac{1}{2}J_0.$$

These equations determine the first approximations to α, η. The remarkable relation between them is modified in the higher approximations.

The integration of the equations (45) will furnish for P, Q values depending on sines, and for K a value depending on cosines of arguments of the form $\beta t + \beta'$.

238. On integration we can add arbitrary constants to δP, δQ, δK: these arbitraries, since the necessary number has been already introduced, may be determined at will. That added to δK merely adds to K_0 and it may therefore be put zero; $K - K_0$ is thus expressed as a series of sines. The constant additive to δP will also be put zero, so that P is expressed as a series of sines. The constant part of Q was $2\sin\frac{1}{2}J_0$, where J_0 was arbitrary; to δQ we add a constant κ, so that Q is expressed in the form

$$2\sin\tfrac{1}{2}J_0 + \kappa + \text{series of cosines}.$$

The constant κ is used so that, when the latitude has been found, the coefficient of the principal term (which has as its argument the distance of the Moon from the node) is $\sin J_0$—this coefficient being determined directly from observation.

A careful investigation of the meanings to be attached to these constants and to those defined in Art. 232, and a comparison with the constants used by Delaunay, is given by Newcomb, *Transformation of Hansen's Lunar Theory**. Another paper, *Investigation of Corrections to Hansen's Tables of the Moon with Tables for their Application*†, by the same author, may also be consulted with advantage.

To find the next Approximation to $n_0 z$, ν, P, Q, K.

239. When the disturbing forces were neglected, we had $z = g$, $\nu = 0$ and P, Q, K constants. Let δz, ν, δP, δQ, δK be the parts, just found, depending on the first order of the disturbing forces. In order to find the next approximation to the values of the variables, it is necessary to substitute in R and in the various functions used, instead of the initial values of the variables, their initial values increased by the parts just found. This is easily done by Taylor's theorem in the following manner.

* *Astron. Papers for the Amer. Ephemeris*, Vol. I. pp. 57–107.

† *Papers published by the Commission on the Transit of Venus*, Pt. III. pp. 1–51.

From the value of δz we deduce, by putting $t=\tau$, that of $\delta\zeta$. Now the expression (18) for dW/dt is, owing to the general form in which the disturbing function has been expressed (Art. 220), a function of z, ζ, ν, P, Q, K. Put $n_0\zeta=\gamma+n_0\delta\zeta$, and let W' be the value of W when $\delta\zeta=0$. We have

$$W=W'+\frac{\partial W'}{\partial\gamma}n_0\delta\zeta+\frac{1}{2!}\frac{\partial^2 W'}{\partial\gamma^2}(n_0\delta\zeta)^2+\ldots;$$

and so for any function containing ζ. Expand the expression (18) for dW/dt, in this way. The factor of y is quite easy to calculate when $\zeta=\gamma$. After putting $y=y_0+\delta y$ (where y_0 is the first approximation to y), the values of W_0, h/h_0, y_0, furnished by the previous approximation, are inserted; the value of δy will be afterwards determined so that no terms proportional to the time shall be present in W. The only difficulty that remains is the calculation of the terms containing $\mathfrak{P}$, $\mathfrak{T}$, and this arises from the presence of ν. It is to be remembered that when $\zeta=\gamma$ we have $\bar{\rho}=\rho_0$, $\bar{\phi}=\phi_0$.

240. Denote the first two terms of dW'/dt by T, so that

$$T=h_0\mathfrak{T}r\left[2\frac{\rho_0}{r}\cos(\bar{f}-\phi_0)-1+2\frac{h^2}{h_0^2}\frac{\rho_0}{a_0(1-e_0^2)}\{\cos(\bar{f}-\phi_0)-1\}\right]+2h_0\frac{\rho_0}{r}\mathfrak{P}r\sin(\bar{f}-\phi_0).$$

Put for $\mathfrak{P}$, $\mathfrak{T}r$ their values $\partial R/\partial r$, $\partial R/\partial\omega$, and let

$$\bar{G}=2h_0\left\{\frac{\rho_0}{\bar{r}}\frac{\partial\bar{R}}{\partial\omega}\cos(\bar{f}-\phi_0)+\frac{\rho_0}{\bar{r}}\bar{r}\frac{\partial\bar{R}}{\partial\bar{r}}\sin(\bar{f}-\phi_0)\right\},$$

$$\bar{U}=2\frac{h_0\rho_0}{a_0(1-e_0^2)}\frac{\partial\bar{R}}{\partial\omega}[\cos(\bar{f}-\phi_0)-1],\qquad \bar{\Sigma}=-h_0\frac{\partial\bar{R}}{\partial\omega},$$

where $\bar{R}$ denotes the value of R (Art. 220) when ν is put zero. Let

$$\begin{aligned}R=R^{(1)}+R^{(2)}+\ldots\quad&=(1+\nu)^2\bar{R}^{(1)}+(1+\nu)^3\bar{R}^{(2)}+\ldots\\&=\bar{R}+(2\nu+\nu^2)\bar{R}^{(1)}+(3\nu+3\nu^2+\nu^3)\bar{R}^{(2)}+\ldots,\end{aligned}$$

so that $\bar{R}^{(1)}, \bar{R}^{(2)}, \ldots$, as well as $\bar{R}$, are independent of ν explicitly.

We have evidently, since $r=\bar{r}(1+\nu)$,

$$r\frac{\partial R^{(1)}}{\partial r}=\bar{r}\frac{\partial\bar{R}^{(1)}}{\partial\bar{r}}(1+\nu)^2,\qquad r\frac{\partial R^{(2)}}{\partial r}=\bar{r}\frac{\partial\bar{R}^{(2)}}{\partial\bar{r}}(1+\nu)^3,\qquad \frac{\partial R^{(1)}}{\partial\omega}=\frac{\partial\bar{R}^{(1)}}{\partial\omega}(1+\nu)^2,\ \text{etc.};$$

and therefore, if the values of $\bar{G}$, $\bar{U}$, $\bar{\Sigma}$ be inserted in T, we obtain

$$\begin{aligned}T^{(1)}&=\bar{G}^{(1)}(1+\nu)+\bar{U}^{(1)}(1+\nu)^2\frac{h^2}{h_0^2}+\bar{\Sigma}^{(1)}(1+\nu)^2\\&=\bar{T}^{(1)}+\nu\bar{G}^{(1)}+\bar{U}^{(1)}\left\{\frac{h^2}{h_0^2}-1+(2\nu+\nu^2)\frac{h^2}{h_0^2}\right\}+\bar{\Sigma}^{(1)}(2\nu+\nu^2)\qquad(46),\end{aligned}$$

where $\bar{T}=\bar{G}+\bar{U}+\bar{\Sigma}$; the numbers in brackets denote that the term $R^{(1)}$ of R is alone considered. A similar expression may be obtained for $\bar{T}^{(2)}$.

Hence $\bar{T}$ is a function of nz, P, Q, K, and it is independent of ν, h/h_0; if $\bar{T}_0$ be its value when δz, δP, δQ, δK are put zero, we have, by Taylor's theorem,

$$\bar{T}=\bar{T}_0+\frac{\partial\bar{T}_0}{\partial g}n_0\delta z+\frac{\partial\bar{T}_0}{\partial P}\delta P+\frac{\partial\bar{T}_0}{\partial Q}\delta Q+\frac{\partial\bar{T}_0}{\partial K}\delta K+\tfrac{1}{2}\frac{\partial^2\bar{T}_0}{\partial g^2}(n_0\delta z)^2+\ldots.$$

In this we substitute the values of δz, δP, δQ, δK furnished by the previous approximation. Since, in (46), $\bar{G}$, $\bar{U}$, $\bar{\Sigma}$ are all multiplied by small quantities, we can substitute elliptic values for the quantities present in the expressions which they denote; ν, h/h_0 receive the values furnished by the previous approximation. As $\bar{T}_0$ is the quantity called T_0 in Art. 226, we already have its value. Hence the terms in (18) can all be expressed in terms of the time when it is desired to obtain the second approximation to W.

In the same way we can isolate ν, h in the equations (45) for P, Q, K. For example, if $n_0 B$ be that portion of the first of these equations which depends on R, we have

$$n_0 B = -h\left(\frac{\partial R}{\partial Q}\cos^2\tfrac{1}{2}J + \tfrac{1}{4}P\frac{\partial R}{\partial K}\right).$$

Put
$$\bar{B} = -\frac{a_0}{\sqrt{1-e_0^2}}\left(\frac{\partial \bar{R}}{\partial Q}\cos^2\tfrac{1}{2}J + \tfrac{1}{4}P\frac{\partial \bar{R}}{\partial K}\right);$$

then
$$B^{(1)} = \bar{B}^{(1)} + \bar{B}^{(1)}\left\{\frac{h}{h_0} - 1 + \frac{h}{h_0}(2\nu + \nu^2)\right\},$$

which can be treated as before.

All the equations are finally expressed in terms of the time and integrated, the determination of δy, δa, $\delta\eta$ being made as in the first approximation.

Reduction to the Instantaneous Ecliptic.

241. A final step is necessary to obtain the longitude and the latitude referred to the ecliptic; the parallax is found from r as in Art. 162.

Draw MH (fig. 9, Art. 217) perpendicular to $X'\Omega$. Then

$$X'H = \text{longitude} = V, \qquad HM = \text{latitude} = U.$$

Also, from the right-angled triangle $MH\Omega$,

$$\sin(V - \psi')\cos U = \sin(v - \psi)\cos J, \quad \sin U = \sin J\sin(v - \psi)\ \ldots(47).$$

From the relations (10), (21),

$$v - \psi = \bar{f} + n_0 yt + \pi_0 + n_0(\alpha - \eta)t + N + K - \pi_0$$
$$= \bar{f} + \omega + \delta N + \delta K = g + \omega + (\bar{f} - g) + \delta N + \delta K,$$

where we now take ω to denote only the mean part of its value given by (29). Also, $\cos J = 1 - (P^2 + Q^2)/2$, and

$$\psi' = -n_0(\alpha + \eta)t - N + K + \pi_0', \qquad \text{by (21)},$$
$$= -n_0(\alpha + \eta)t - N_0 + K_0 + \delta N + \delta K$$
$$= \mu_1 - \delta N + \delta K,$$

where μ_1 denotes the mean longitude of the node on the ecliptic. Thus $g + \omega + \mu_1$ denotes the mean longitude of the Moon on the ecliptic. Since $\bar{f} - g$, δN, δK, P, Q contain only periodic terms which are at least of the first order of small quantities, we can, after substituting their values, obtain V and U in the usual form.

Hansen finds U from the equation

$$\sin U = \sin J_0 \sin(\bar{f} + \omega) + s,$$

so that s denotes the perturbations of the sine of the latitude; $\bar{f}$ may be expanded in powers of $n_0\delta z$ as in Art. 232.

For the details of the transformations, the reader is referred to the *Darlegung* I. § 9. They are also to be found in Tisserand, *Méc. Cél.* Vol. III., Arts. 153—155 of Chap. XVII.: the chapter referred to consists of an account of Hansen's *Darlegung*.

CHAPTER XI.

METHOD WITH RECTANGULAR COORDINATES.

242. The use of rectangular coordinates is an essential feature of the latest method for the treatment of the solar inequalities in the Moon's motion. The equations of motion, referred to axes of which two move in their own plane with uniform angular velocity while the third is fixed, have already been investigated in Section (iii) of Chapter II.; a plan for the complete solution of these equations by means of series will now be given. The method of obtaining it is, to a certain extent, one of continued approximation. The approximations do not, however, proceed according to powers of the disturbing forces, that is of m, but according to powers of $e, e', \gamma, 1/a'$—the constants which are naturally present in the coefficients but which, in the earlier approximations, do not occur explicitly in the arguments. The advantage of the method used here is due to the possibility of carrying a coefficient to any degree of accuracy, as far as m is concerned, without making the large number of calculations which the methods discussed earlier would entail; a reference to Art. 154 will show the importance of this.

The theory is adapted to a complete literal development, but the labour necessary to secure accurate expressions for the coordinates of the Moon will be best employed by giving to m its numerical value and by leaving the other constants arbitrary. The fact that the mean motions of the Sun and Moon are the two constants which have been obtained by observation with the greatest degree of accuracy, will justify this abbreviation of the work; any alteration in their values which future observations might give, must necessarily be very minute, and its effect can therefore be deduced from the literal developments of Delaunay.

A difficulty which caused some trouble in de Pontécoulant's theory does not arise here. In obtaining the approximations to a given order, it was frequently necessary to consider terms of orders higher than those actually required, owing to the presence of small divisors. Since the order of a term, as far as $e, e', \gamma, 1/a'$ are concerned, is never lowered by integration, this

proceeding will be unnecessary here, for the method enables us to include all powers of m when calculating terms of a given order with respect to the four constants just mentioned: those terms, whose orders with respect to m are lowered by the integrations, merely present themselves with larger coefficients than they might otherwise be expected to have.

243. The introduction of this method is due to Dr G. W. Hill who in 1877 published two important papers entitled 'Researches in the Lunar Theory*' and 'On the Part of the Motion of the Lunar Perigee which is a function of the Mean Motions of the Sun and Moon†'. These papers, besides throwing a new light on the methods of celestial mechanics, contain entirely new analytical devices for treating the problem of three bodies; M. Poincaré, in the preface to the first volume of his *Mécanique Céleste*, remarks, 'Dans cette œuvre...il est permis d'apercevoir le germe de la plupart des progrès que la Science a faits depuis.'

The first paper is devoted partly to an examination of the equations most useful for the actual determination of the Moon's motion and of the limits between which the radius vector must lie, and partly to the determination of the principal parts of those inequalities in the motion which have been called, in Art. 166, the Variational Class. In the second paper, the determination of $(1-c)n$, the principal part of the motion of the perigee, is shown to depend on the computation of an infinite determinant and the numerical value of this quantity is calculated with a high degree of accuracy.

On the publication of these papers, the late Prof. J. C. Adams gave the results of an investigation which he had made several years before, in order to find the corresponding part of the motion of the node‡. This likewise depends on the calculation of an infinite determinant of similar form; owing, however, to the simplicity of the equation from which it arises, no transformations, like those necessary in the case of the perigee, are required. The full details of his investigation have not been published.

The further developments which have been made and which directly concern the lunar theory, will be referred to in the course of the Chapter.

The *limitations imposed on the problem* are the same as those adopted in the methods of de Pontécoulant and Delaunay. The disturbing function used is that of Art. 8, and the orbit of the Sun is an ellipse situated in the plane of (XY) with the Earth occupying one focus.

The preliminary Limitations imposed on the Equations of Motion. The Intermediate Orbit.

244. The general equations of motion (17), (18) of Chap. II., referred to axes moving with uniform angular velocity n' in the plane of reference, have undergone certain transformations: the forms to be used are there

* *Amer. Journ. Math.* Vol. I. pp. 5–26, 129–147, 245–260.

† Cambridge U.S.A. 1877 and *Acta Math.* Vol. VIII. pp. 1–36.

These two papers will be respectively referred to below by the titles 'Researches' and 'Motion of the Perigee' and by the pages of the journals in which they appeared.

‡ See footnote, p. 230.

numbered (23), (19), (18). Since it is not possible to solve these equations directly, it will be necessary to neglect certain terms which are known to be small. Connected with this limitation is the choice of an intermediate orbit: the intermediary will not be the ellipse or the modified ellipse chosen by previous lunar theorists but will be defined to be a certain periodic solution of the differential equations after some, but not all, of those parts of them due to the Sun's action have been neglected. It is assumed, as before, that expansions, in powers and products of the small quantities which will be initially neglected in the differential equations, are possible.

Let the equations (23), (19), (18) of Chap. II. be limited by neglecting the small quantities e', $1/a'$. Then $r' = a'$, $rS = X$ and (Art. 19) $\Omega = 0$. Further, neglect the latitude of the Moon, so that $z = 0$. The equation (18) disappears and the equations (23), (19) reduce to

$$\left.\begin{aligned} D^2(\upsilon\sigma) - D\upsilon D\sigma - 2\mathrm{m}(\upsilon D\sigma - \sigma D\upsilon) + \tfrac{9}{4}\mathrm{m}^2(\upsilon+\sigma)^2 = C,\\ D(\upsilon D\sigma - \sigma D\upsilon - 2\mathrm{m}\upsilon\sigma) + \tfrac{3}{2}\mathrm{m}^2(\upsilon^2 - \sigma^2) = 0 \end{aligned}\right\} \quad\ldots\ldots\ldots (1).$$

It is advisable to notice that the equations (1) are equivalent to equations (17) of Art. 19 with $\Omega = 0$ and *to no others*. The second of equations (1) may be written

$$\{D^2\upsilon + 2\mathrm{m}D\upsilon + \tfrac{3}{2}\mathrm{m}^2(\upsilon+\sigma)\}/\upsilon = \{D^2\sigma - 2\mathrm{m}D\sigma + \tfrac{3}{2}\mathrm{m}^2(\upsilon+\sigma)\}/\sigma = \chi,$$

suppose. The first of them is then

$$2\upsilon\sigma\chi = C - D\upsilon D\sigma - \tfrac{3}{4}\mathrm{m}^2(\upsilon+\sigma)^2.$$

Whence, by differentiation,

$$2\upsilon\sigma D\chi + 2\chi D(\upsilon\sigma) = -D^2\upsilon D\sigma - D^2\sigma D\upsilon - \tfrac{3}{2}\mathrm{m}^2(\upsilon+\sigma)(D\upsilon + D\sigma) = -\chi(\upsilon D\sigma + \sigma D\upsilon) = -\chi D(\upsilon\sigma).$$

Therefore $$\frac{D\chi}{\chi} + \frac{3}{2}\frac{D(\upsilon\sigma)}{\upsilon\sigma} = 0 \quad \text{or,} \quad \text{Const.} = \chi(\upsilon\sigma)^{\frac{3}{2}} = \chi r^3,$$

which proves the equivalence of the two forms.

The constant κ, which has disappeared, must be determined in terms of the arbitraries by a reference to one of the original equations of motion. See Art. 21.

In order to see the connection between these equations and the ordinary forms by which the lunar motion is expressed, reference is made to the expressions collected in Art. 149. When $e' = 0$, $1/a' = 0$, $z = 0$ (or, in the notation of Chap. VII., $e' = 0$, $a/a' = 0$, $\gamma = 0$), the coefficients of the remaining periodic terms depend on m, e only, while the arguments depend only on the angles $2\xi = 2(n - n')t + 2(\epsilon - \epsilon')$, $\phi = cnt + \epsilon - \varpi$. Hence, *the equations* (1) *will furnish all inequalities which depend only on* m *and* e and will besides give the value of c so far as this quantity depends on m, e.

245. The *Intermediate Orbit* is defined to be the path described by the Moon when we neglect e', a/a', γ, e; it therefore consists of those terms in the expression for the Moon's motion which depend only on m. Now e is

an arbitrary of the solution of the equations which determine the Moon's motion, and since the equations (1) determine all inequalities which depend only on m, e, the intermediate orbit must be a solution of (1) not containing the full number of arbitrary constants.

The inequalities which depend only on m have been considered in case i. of Chap. VII. In Chap. VIII. (Art. 166) they have been more fully examined; it was seen that they correspond to a symmetrical closed curve referred to axes moving with angular velocity n' about the Earth: this closed curve was called the Variational curve. Since the coordinates in equations (1) refer to axes moving in the same manner, the Intermediate Orbit is the Variational curve. If we look at the constants present in the expressions for the coordinates of a point on the Variational curve, it is seen that only two of them are arbitrary, namely, n, ϵ, for a was defined by the equation $\mu = n^2a^3$. *Hence the Intermediate Orbit is a periodic solution of* (1) *involving only two arbitrary constants.* The first object in view is the determination of this orbit in the most effective and accurate manner.

246. The only small quantity not neglected in the equations (1) is now n' or m. The reason for retaining this small quantity at the outset is derived from a knowledge of previous results. When the full literal developments in a theory like that of Delaunay are examined, it is seen directly that if the series which represents any coefficient converges slowly, the slow convergence takes place chiefly when the series is arranged according to powers of m and not when it is arranged according to powers of e, e', a/a' or γ. It is therefore of the greatest importance to find an intermediate orbit in which the coefficients may be obtained with any accuracy desirable as far as m is concerned.

It might have seemed better to consider the complete solution of the equations (1) as the intermediate orbit; this evidently involves a determination of c. Later developments have shown that there is a saving of labour if we find those terms which depend on m, e' immediately after obtaining the parts which depend on m only, the next step being the determination of the inequalities which depend on m, e', e. This plan will not be followed here, because the developments necessary to show the reason for it are too long to be inserted in this Chapter (see Art. 288); the order of procedure will be the same as that of Chap. VII.

It is evident that if we neglect m in the equations (1), they must reduce to those for elliptic motion. Since $v\sigma = r^2$, $-(n-n')^2 Dv D\sigma = (\text{vel.})^2$, the first of them can, in this case, be deduced by eliminating μ/r between equations (2), (3) of Art. 12, after putting R, z zero. The second equation expresses the fact that equal areas are described in equal times.

(i) The Terms whose Coefficients depend only on m. The Determination of the Intermediate Orbit.

247. In order to arrive at the forms of the expressions for v, σ when the Intermediate Orbit is under consideration, recourse may be had to Art. 166. It was there seen that when the terms dependent on m are alone retained, r,

the radius vector, and v, the longitude measured from a fixed line, are given by

$$a/r = \Sigma b_{2i} \cos 2i\xi, \qquad v - n't - \epsilon' = \xi + \Sigma b'_{2i} \sin 2i\xi, \qquad (i = 0, 1, \ldots \infty).$$

Here $\xi = (n - n')t + \epsilon - \epsilon'$ and b_{2i}, b'_{2i} depend only on m. But since X, Y are the coordinates referred to axes, of which that of X points towards the mean place of the Sun, we have

$$X = r \cos (v - n't - \epsilon'), \qquad Y = r \sin (v - n't - \epsilon') \ldots\ldots\ldots(2).$$

Hence X, Y are of the forms

$$X = \sum_{i=0}^{\infty} A_{2i} \cos (2i+1)\xi, \quad Y = \sum_{i=0}^{\infty} A'_{2i} \sin (2i+1)\xi.$$

Let $\qquad A_{2i} = \mathrm{a}\,(a_{2i} + a_{-2i-1}), \qquad A'_{2i} = \mathrm{a}\,(a_{2i} - a_{-2i-1}),$

where i receives negative as well as positive values and where we suppose $a_0 = 1$; a_0 will, however, be retained for symmetry. We have then

$$\left.\begin{aligned} X &= \mathrm{a}\Sigma_i a_{2i} \cos (2i+1)\xi, \\ Y &= \mathrm{a}\Sigma_i a_{2i} \sin (2i+1)\xi, \end{aligned}\right\} (i = -\infty \ldots + \infty) \ldots\ldots\ldots\ldots(2').$$

Whence, since $v = X + Y\sqrt{-1}$, $\sigma = X - Y\sqrt{-1}$, the forms to be given to v, σ are

$$v = \mathrm{a}\Sigma_i a_{2i} \exp.(2i+1)\xi\sqrt{-1}, \qquad \sigma = \mathrm{a}\Sigma_i a_{2i} \exp.(-2i-1)\xi\sqrt{-1}.$$

Two arbitrary constants n, t_0 (Arts. 18, 21) have been introduced into equations (1). Let n have the same meaning as before, namely, the observed mean motion of the Moon, and let

$$(n - n')t_0 = -(\epsilon - \epsilon');$$

then
$$(n - n')(t - t_0) = (n - n')t + \epsilon - \epsilon' = \xi,$$
$$\mathrm{m} = n'/(n - n') = m/(1 - m).$$

As ϵ, n were arbitraries of the solution previously found, n, t_0 are thus defined to be the arbitraries of that solution of (1) which is considered here.

Finally, since $\zeta = \exp.(n - n')(t - t_0)\sqrt{-1}$, (Art. 18), the assumed values for v, σ may, after putting $-i-1$ for i in the expression for σ, be written*

$$v = \mathrm{a}\Sigma_i a_{2i}\zeta^{2i+1}, \qquad \sigma = \mathrm{a}\Sigma_i a_{-2i-2}\zeta^{2i+1} \ldots\ldots\ldots\ldots\ldots\ldots(3);$$

or,
$$v\zeta^{-1} = \mathrm{a}\Sigma_i a_{2i}\zeta^{2i}, \qquad \sigma\zeta = \mathrm{a}\Sigma_i a_{-2i}\zeta^{2i} \ldots\ldots\ldots\ldots\ldots\ldots(3');$$

where $i = -\infty \ldots + \infty$.

It is now only a question of so determining the coefficients a_{2i} that these values for v, σ may satisfy the equations (1).

* In the *Researches* p. 130, a_i takes the place of the coefficient here called $\mathrm{a}a_{2i}$. The change is convenient, firstly, because the chief consideration is the determination of $\mathrm{a}_i/\mathrm{a}_0$ and secondly, because it will not now be necessary to introduce another letter for the coefficients of the parallactic terms (Art. 277).

248. It must be remembered that the object in view is the determination of the coefficients a_{2i}. They have been obtained to the order m^4 in Chap. VII., but no general law was forthcoming by which they could be found to any degree of accuracy without great labour. We have only made use of the results of that Chapter in order to discover the *form* of the required solution of equations (1). From the point of view of the lunar theory, it is very important that the connections between the results of the various theories should be clearly set forth; for this reason it has seemed preferable to deduce the forms of the solutions from our previous knowledge rather than to obtain them by a general consideration of the differential equations.

The latter method has been followed by Poincaré in his *Mécanique Céleste*. In Chap. III., Vol. I. of his treatise, he has considered the conditions under which the equations of dynamics admit of periodic solutions. In §§ 39—41, he gives some applications of his results to the Problem of Three Bodies and, as a special case, he proves that the equations (1) or rather the equations (25) of Art. 23 above, from which the former were deduced, do admit of periodic solutions in general. Moreover, in the same Chapter, he shows that these solutions, for sufficiently small values of the parameter, are in general developable in series.

The Determination of the Coefficients of the Variational Inequalities.

249. Let j be an integer with the same range of values as i, namely, from $+\infty$ to $-\infty$. Since $D\zeta^i = i\zeta^i$, we have, from the equations (3),

$$Dv = \mathrm{a}\Sigma_i (2i+1)\, a_{2i}\zeta^{2i+1}, \qquad D\sigma = \mathrm{a}\Sigma_i (2i+1)\, a_{-2i-2}\zeta^{2i+1},$$

$$v\sigma = \mathrm{a}^2 \left[\Sigma_i a_{2i}\zeta^{2i+1}\right]\left[\Sigma_j a_{-2j-2}\zeta^{2j+1}\right] = \mathrm{a}^2\Sigma_i\Sigma_j a_{2i}a_{-2j-2}\zeta^{2i+2j+2}.$$

As j and i extend independently from $+\infty$ to $-\infty$, we may put $j-i-1$ for j. The expression for $v\sigma$ then becomes

$$v\sigma = \mathrm{a}^2\Sigma_j\Sigma_i a_{2i}a_{2i-2j}\zeta^{2j}.$$

Similarly,

$$v^2 = \mathrm{a}^2\Sigma_j\Sigma_i a_{2i}a_{2j-2i-2}\zeta^{2j}, \qquad \sigma^2 = \mathrm{a}^2\Sigma_j\Sigma_i a_{2i}a_{-2j-2i-2}\zeta^{2j},$$

$$Dv\,.\,D\sigma = -\,\mathrm{a}^2\Sigma_j\Sigma_i (2i+1)(2i-2j+1)\, a_{2i}a_{2i-2j}\zeta^{2j},$$

$$vD\sigma - \sigma Dv = -\,\mathrm{a}^2\Sigma_j\Sigma_i (4i-2j+2)\, a_{2i}a_{2i-2j}\zeta^{2j}.$$

When these results have been substituted in equations (1), the coefficients of the several powers of ζ must be equated to zero in order that the values assumed for v, σ may satisfy the equations. The coefficients of ζ^{2j} give

$$\left.\begin{aligned} \Sigma_i\left[4j^2 + (2i+1)(2i-2j+1) + 4(2i-j+1)\,\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2\right] a_{2i}a_{2i-2j} \qquad\qquad & \\ + \tfrac{9}{4}\mathrm{m}^2\Sigma_i a_{2i}\left(a_{2j-2i-2} + a_{-2j-2i-2}\right) = 0, & \\ 4j\Sigma_i (2i-j+1+\mathrm{m})\, a_{2i}a_{2i-2j} - \tfrac{3}{2}\mathrm{m}^2\Sigma_i a_{2i}\left(a_{2j-2i-2} - a_{-2j-2i-2}\right) = 0 & \end{aligned}\right\} \ldots (4);$$

except for the value $j=0$, when the second equation is identically satisfied and the right hand side of the first equation is C/a^2.

Multiply the second of these equations by $(2m+1)/2j$ and subtract from

the first; also, divide the second equation by $4j$. Excluding the value $j=0$, the results are

$$\left.\begin{aligned}
0 &= \Sigma_i \{4j^2 - 1 - 2\mathrm{m} + \tfrac{1}{2}\mathrm{m}^2 + 4i^2 - 4ij\}\, a_{2i}a_{2i-2j} \\
&\quad + \left\{\tfrac{9}{4}\mathrm{m}^2 + \tfrac{3}{4}\frac{\mathrm{m}^2}{j}(2\mathrm{m}+1)\right\} \Sigma_i a_{2i}a_{2j-2i-2} + \left\{\tfrac{9}{4}\mathrm{m}^2 - \tfrac{3}{4}\frac{\mathrm{m}^2}{j}(2\mathrm{m}+1)\right\}\Sigma_i a_{2i}a_{-2j-2i-2}, \\
0 &= \Sigma_i (1 - j + \mathrm{m} + 2i)\, a_{2i}a_{2i-2j} - \tfrac{3}{8}\frac{\mathrm{m}^2}{j}\Sigma_i (a_{2i}a_{2j-2i-2} - a_{2i}a_{-2j-2i-2})
\end{aligned}\right\} \text{(4')}.$$

For the case $j=0$ we have, from the first of equations (4),

$$C/\mathrm{a}^2 = 4\Sigma_i \{(2i+1+2\mathrm{m})^2 + \tfrac{1}{2}\mathrm{m}^2\}\, a_{2i}^2 + \tfrac{9}{2}\mathrm{m}^2 \Sigma_i a_{2i}a_{-2i-2} \quad \text{.........(5)};$$

an equation which serves to determine C when a, a_{2i} have been found.

250. As $a_0 = 1$, these equations show that the coefficients a_{2i} are functions of m only. Also, it is not difficult to see that a_{2i} will be of the order $m^{|2i|}$ *. For example, put $j=1$ in equations (4′), and write down a few terms given by small values of i. We obtain

$$\begin{aligned}
0 = {} & (3 - 2\mathrm{m} + \tfrac{1}{2}\mathrm{m}^2)(a_2a_0 + a_0a_{-2}) + (11 - 2\mathrm{m} + \tfrac{1}{2}\mathrm{m}^2)(a_4a_2 + a_{-2}a_{-4}) + \dots \\
& + (3\mathrm{m}^2 + \tfrac{3}{2}\mathrm{m}^3)(a_0^2 + 2a_2a_{-2} + 2a_4a_{-4} + \dots) + (\tfrac{3}{2}\mathrm{m}^2 - \tfrac{3}{2}\mathrm{m}^3)(a_{-2}^2 + 2a_0a_{-4} + 2a_2a_{-6} + \dots), \\
0 = {} & \dots + (4+\mathrm{m})\,a_4a_2 + (2+\mathrm{m})\,a_2a_0 + \mathrm{m}a_0a_{-2} + (-2+\mathrm{m})\,a_{-2}a_{-4} + \dots \\
& - \tfrac{3}{8}\mathrm{m}^2(a_0^2 + 2a_2a_{-2} + 2a_4a_{-4} + \dots - a_{-2}^2 - 2a_0a_{-4} - 2a_2a_{-6} - \dots).
\end{aligned}$$

If we suppose a_4, a_{-4} of higher order than a_2, a_{-2}, these equations show that a_2, a_{-2} are of the order m^2 at least. We may similarly treat the equations for $j=2$ and deduce the fact that a_4, a_{-4} are of the order m^4, and so on. It will appear presently that, in finding the a_{2i} from equations (4′), m cannot be a factor of any of the denominators. We see further that the equations always occur in pairs and that those of principal importance in finding a_{2j}, a_{-2j}, are obtained by equating the coefficients of ζ^{2j} to zero.

251. It is now necessary to show how the coefficients may be most suitably obtained from equations (4′). Owing to the fact that a_{2j} is of the order $|2j|$ at least, they may be found by continued approximation; but in order to explain how the approximations are carried out, some further remarks must be made. Suppose that it be desired to determine a_{2j} correctly to the order $|2j|$, the values of the coefficients, for values of j numerically smaller than that considered, being supposed known. The equations (4′) show that we shall have two simultaneous linear equations for $a_{2j}a_0$, $a_{-2j}a_0$ from which these quantities may be determined; the same is true when we desire to obtain them to any degree of approximation whatever. When we use the numerical value of m at the outset, these are the simplest equations to deal with; but when a literal development in powers of m is required, the labour will be lessened if, before commencing the calculations, we deduce from the two equations giving a_{2j}, a_{-2j} two new equations in which the coefficients of $a_{-2j}a_0$, $a_{2j}a_0$ are respectively zero.

* If a be any real quantity, $|a|$ denotes that a is to be taken positively whichever sign it may have.

The terms containing $a_{2j}a_0$, $a_{-2j}a_0$ in equations (4′) are obtained by putting $i=j$, $i=0$. They are, in the respective equations,

$$(4j^2-1-2\mathrm{m}+\tfrac{1}{2}\mathrm{m}^2)(a_0a_{2j}+a_0a_{-2j}),$$

$$(1+j+\mathrm{m})\,a_0a_{2j}+(1-j+\mathrm{m})\,a_0a_{-2j}.$$

If, therefore, we multiply the two equations by

$$j-1-\mathrm{m}, \qquad 4j^2-1-2\mathrm{m}+\tfrac{1}{2}\mathrm{m}^2 \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

respectively, and add the results, the coefficient of a_0a_{-2j} will be zero and that of $-a_0a_{2j}$ will become

$$-2j(4j^2-1-2\mathrm{m}+\tfrac{1}{2}\mathrm{m}^2)\ldots\ldots\ldots\ldots\ldots\ldots(6').$$

Further, if we divide the resulting equation by this expression, the coefficient of a_0a_{2j} will be -1.

Multiply then the equations (4′) by the two expressions (6), respectively, add them and divide the resulting equation by (6′). Let it be written

$$\Sigma_i\{[2j,\,2i]\,a_{2i}a_{2i-2j}+[2j,]\,a_{2i}a_{2j-2i-2}+(2j,)\,a_{2i}a_{-2j-2i-2}\}=0 \;\ldots\ldots (7).$$

The coefficients $[2j,\,2i]$, $[2j,]$, $(2j,)$ * will be

$$\left.\begin{aligned}
&[2j,2i]=\\
&\frac{\{(4j^2-1-2\mathrm{m}+\frac{1}{2}\mathrm{m}^2)+4i(i-j)\}(j-1-\mathrm{m})+(4j^2-1-2\mathrm{m}+\frac{1}{2}\mathrm{m}^2)\{2i-(j-1-\mathrm{m})\}}{-j\,(8j^2-2-4\mathrm{m}+\mathrm{m}^2)}\\
&\qquad=-\frac{i}{j}\,\frac{8j^2-2-4\mathrm{m}+\mathrm{m}^2+4\,(i-j)(j-1-\mathrm{m})}{8j^2-2-4\mathrm{m}+\mathrm{m}^2};\\
&[2j,]=\frac{\{\frac{9}{4}j\mathrm{m}^2+\frac{3}{4}\mathrm{m}^2(2\mathrm{m}+1)\}(j-1-\mathrm{m})-\frac{3}{8}\mathrm{m}^2(4j^2-1-2\mathrm{m}+\frac{1}{2}\mathrm{m}^2)}{-j^2(8j^2-2-4\mathrm{m}+\mathrm{m}^2)}\\
&\qquad=-\frac{3}{16}\,\frac{\mathrm{m}^2}{j^2}\,\frac{4j^2-8j-2-4\,(j+2)\,\mathrm{m}-9\mathrm{m}^2}{8j^2-2-4\mathrm{m}+\mathrm{m}^2};\\
&(2j,)=\frac{\{\frac{9}{4}j\mathrm{m}^2-\frac{3}{4}\mathrm{m}^2(2\mathrm{m}+1)\}(j-1-\mathrm{m})+\frac{3}{8}\mathrm{m}^2(4j^2-1-2\mathrm{m}+\frac{1}{2}\mathrm{m}^2)}{-j^2(8j^2-2-4\mathrm{m}+\mathrm{m}^2)}\\
&\qquad=-\frac{3}{16}\,\frac{\mathrm{m}^2}{j^2}\,\frac{20j^2-16j+2-4\,(5j-2)\,\mathrm{m}+9\mathrm{m}^2}{8j^2-2-4\mathrm{m}+\mathrm{m}^2}
\end{aligned}\right\}$$

$$\ldots\ldots\ldots(7').$$

It is evident that $[2j,\,0]=0$, $[2j,\,2j]=-1$, so that the coefficients of a_0a_{-2j}, a_0a_{2j} are respectively 0, -1.

Equation (7) will also serve for the determination of a_{-2j}. For j may receive both positive and negative values and, if $-j$ is put for j, the coeffi-

* In the *Researches*, p. 134, 135, these are denoted by $[j,\,i]$, $[j]$, (j) respectively. The change facilitates the determination of the terms dependent on e, e', etc.

cient of $a_0 a_{2j}$ becomes $[-2j, 0]$ which is zero, while that of $a_0 a_{-2j}$ is $[-2j, -2j]$ which is equal to -1. The value $j=0$ is excluded as before.

252. The denominator (6′) has its least value when $j=\pm 1$ and it cannot then vanish for any real value of m. Also, since $8j^2-2$ is never zero, m cannot appear as a divisor: the statement made in Art. 250, with reference to the order of a_{2i}, will therefore be true for all these coefficients.

It will be noticed that we have obtained one equation (7) instead of the two (4′): this is a necessary consequence of the use of imaginary variables. With such variables it is possible in general to replace two equations by one, as was seen in Art. 18, where either of the two equations (14) of that article was a complete substitute for the first two of those there numbered (12). The equations (4) of Art. 249 do not possess this property because they arise from the equations (1) which respectively contain real terms and imaginary terms only (see Art. 21). When, however, the equations (4) are combined in a suitable way, one equation can be made to replace the two. In either of the equations (4), positive values of j are sufficient; this can be seen by putting $-j$ for j and then changing i into $i-j$ in the first summations: after the two changes the equations become the same as before.

253. *The determination of the coefficients a_{2i} from the equations* (7).

In order to show the method of finding the coefficients a_{2i} from these equations, the latter will be written out in full with a number of terms sufficient to find a_2, a_{-2}, a_6, a_{-6} to the ninth order inclusive and a_4, a_{-4} to the seventh order inclusive. In writing them down, it is only necessary to recall that $[2j,]$, $(2j,)$ are of the second order and that a_{2j} is of the order $|2j|$. In the parts multiplied by $[2j,]$, $(2j,)$, a term $a_{2i}a_{2i'}$ occurs twice when i, i' are different. We find

$$a_0 a_2 = [2, -2]\, a_{-2}a_{-4} + [2, 4]\, a_4 a_2 + [2,]\,(a_0^2 + 2a_{-2}a_2) + (2,)\,(a_{-2}^2 + 2a_0 a_{-4}),$$

$$a_0 a_{-2} = [-2, -4]\, a_{-4}a_{-2} + [-2, 2]\, a_2 a_4 + [-2,](a_{-2}^2 + 2a_0 a_{-4}) + (-2,)\,(a_0^2 + 2a_{-2}a_2);$$

$$a_0 a_4 = [4, 2]\, a_2 a_{-2} + [4,]\,(2a_0 a_2),$$

$$a_0 a_{-4} = [-4, -2]\, a_{-2}a_2 + (-4,)\,(2a_0 a_2);$$

$$a_0 a_6 = [6, 2]\, a_2 a_{-4} + [6, 4]\, a_4 a_{-2} + [6,]\,(a_2^2 + 2a_0 a_4),$$

$$a_0 a_{-6} = [-6, -4]\, a_{-4}a_2 + [-6, -2]\, a_{-2}a_4 + (-6,)\,(a_2^2 + 2a_0 a_4).$$

These are easily solved by continued approximation. Since $a_0 = 1$, we have, neglecting terms of the sixth and higher orders,

$$a_2 = [2,], \quad a_{-2} = (-2,), \qquad \text{to } \mathrm{m}^5.$$

Whence, neglecting terms of the eighth and higher orders,

$$a_4 = [4, 2][2,](-2,) + 2\,[4,][2,], \quad a_{-4} = [-4, -2](-2,)[2,] + 2\,(-4,)[2,], \quad \text{to } \mathrm{m}^7.$$

Similarly, using these results, we can obtain a_6, a_{-6} correctly to the order m^9.

To obtain a_2, a_{-2} correctly to the order m^9, it is sufficient to add to the values of a_2, a_{-2} just found, the terms previously neglected; these latter are obtained to the required accuracy if we calculate them with the values of a_2, a_{-2}, a_4, a_{-4} previously found. For a_{-2}, a_{-4}, which are of the second and fourth orders, were respectively obtained correctly to the fifth and seventh orders: their product is therefore correct to the ninth order; the same property holds for the other terms. Should it be desired to obtain their values still more approximately, a similar process will serve. We proceed by continued approximation, using the results of the previous approximation for the calculation of the terms previously neglected.

If these results be utilised to obtain a numerical development, we can, at the outset, calculate the coefficients $[2j, 2i]$, $[2j,]$, $(2j,)$ for all values of j, i required; the continued approximations are then very simple. For a literal development in powers of m, these coefficients may be first expanded in powers of m to the degree of accuracy ultimately demanded; the approximations then only entail multiplications of series of such powers.

254. The rapidity with which the values approximate may be grasped from the fact that each new approximation carries the value of the coefficient under consideration *four* orders higher. Also, for each such increase of accuracy, it is only necessary to add four new terms to the equation for that coefficient,—two with factors of the form $[2j, 2i]$, one with the factor $2[2j,]$ and one with the factor $2(2j,)$. The advantage of the method, when we compare it with a laborious one like that of de Pontécoulant, is very striking. Dr Hill has calculated the values of a_{2i} literally* to the order m^9 and numerically† to fifteen places of decimals. To show the rapidity with which they approximate in the latter case, the value of a_{-2}—the largest coefficient—is given below.

He takes $n'/(n-n') = m = \cdot 08084\ 89338\ 08312$ and finds

1st Approx.	$= -\cdot 00869$	58084	99634,
additional terms in the 2nd „	$= +\cdot 00000$	00615	51932,
„ „ „ 3rd „	$= -\cdot 00000$	00000	13838,
resulting value of a_{-2}	$= -\cdot 00869$	57469	61540.

When literal developments in powers of m are considered, the convergency of the series obtained depends to a large extent on the system of divisors $2(4j^2-1)-4m+m^2$. These divisors increase very rapidly with j. The smallest of them, given by $j=\pm 1$, is $6-4m+m^2$; slow convergence would then chiefly arise from this divisor. Dr Hill inquires what function of m, of the form $\overline{m}=m/(1+am)$, will make the expansion, in powers of $\overline{m}$, of the inverse of this divisor converge most quickly. It is easily found that the necessary value for a is $-\frac{1}{3}$; the divisor $6-4m+m^2$ gives rise to the divisor $6+\frac{1}{3}\overline{m}^2$, and new divisors, which are powers of $1+\frac{1}{3}\overline{m}$, are also introduced. The expansion of the inverse of each of these converges quickly.

It may be remarked that it is not necessary to repeat the whole series of approximations in order to have the results expressed in terms of $\overline{m}$. In the results expressed in terms of m, we simply put $m=\overline{m}/(1-\frac{1}{3}\overline{m})$ and then expand in powers of $\overline{m}$, stopping at that power

* *Researches*, pp. 142, 143. † *Id.* pp. 247, 248.

of $\overline{\text{m}}$ to which the expansions in m were carried. Any of the results, whether they have been expressed in polar or rectangular coordinates, may be treated thus.

One of the most interesting parts of the Researches (pp. 250—260) is that which contains the investigation of the forms of the variational curves for increasing values of m. When m is much greater than $\frac{1}{3}$, it is found to be no longer possible to use the expansion in powers of m and mechanical quadratures must be employed. See Art. 166 above.

255. *Determination of* a.

Since κ (or μ) has disappeared from the equations used above, there must be relations connecting κ, μ with n, a. It will now be shown that these relations are of the forms, $\kappa = \text{a}^3\,(1 + \text{powers of m})$, $\mu = n^2\text{a}^3\,(1 + \text{powers of m})$. The first of equations (17) of Art. 19, with $\Omega = 0$, will be used. This equation may be written

$$\frac{\kappa v}{(v\sigma)^{\frac{3}{2}}} = (D^2 + 2\text{m}D + \tfrac{3}{2}\text{m}^2)\,v + \tfrac{3}{2}\text{m}^2\,\sigma.$$

Substitute the values (3) of v, σ. Since the result must hold for all values of ζ, we can, after the substitution, put $\zeta = 1$. We have then

$$v = \sigma = \text{a}\Sigma_i a_{2i}, \quad Dv = \text{a}\Sigma_i\,(2i+1)\,a_{2i}, \quad D^2v = \text{a}\Sigma_i\,(2i+1)^2\,a_{2i};$$

and the result is

$$\kappa\text{a}^{-3}\,(\Sigma_i a_{2i})^{-2} = \Sigma_i\,\{(2i+1+\text{m})^2 + 2\text{m}^2\}\,a_{2i}.$$

But $\kappa = \mu/(n-n')^2 = \mu\,(1+\text{m})^2/n^2$, by Art. 19. Hence

$$\text{a} = \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}(1+\text{m})^{\frac{2}{3}}\,[\Sigma_i\,\{(2i+1+\text{m})^2 + 2\text{m}^2\}\,a_{2i}]^{-\frac{1}{3}}\,[\Sigma_i a_{2i}]^{-\frac{2}{3}} \ldots (8).$$

When the values of the coefficients a_{2i} have been inserted, a will be expressed in the required form. The quantity μ/n^2 is that usually called a^3 in the lunar theory. Hence a/a differs from unity by powers of m only, and it is equal to unity when $\text{m} = 0$.

Since the parallax is observed directly, it will not be generally necessary to make the transformation from a to a in the coefficients; if we desire, for the sake of comparison, to have the results expressed in terms of a, the transformation can always be delayed till the end of the investigation.

The value of a may be also found by substituting for v, σ as before and equating the coefficients of ζ^1 to zero. The results, obtained by this method and by that just given in detail, will be entirely different in form and their agreement will therefore constitute a useful verification.

The value of C may be found (if it be required) from (5). Another method for obtaining it, is given by substituting the values of v, σ in the first of equations (1) and putting $\zeta = 1$ in the result. For this we have

$$v = \sigma = \text{a}\Sigma_i a_{2i}, \quad Dv = -D\sigma = \text{a}\Sigma_i\,(2i+1)\,a_{2i}, \quad D^2v = D^2\sigma = \text{a}\Sigma_i\,(2i+1)^2\,a_{2i}.$$

The two results for C, being also of different forms, furnish another means of verification.

256. *Transformation to real rectangular and to polar coordinates.*

The coefficients $\mathrm{a}a_{2i}$ having been found, the coordinates may be expressed in the usual manner by the formulæ,

$$r\cos(v-nt-\epsilon)=r\cos(v-n't-\epsilon'-\xi)=\tfrac{1}{2}(v\zeta^{-1}+\sigma\zeta),$$
$$r\sin(v-nt-\epsilon)=r\sin(v-n't-\epsilon'-\xi)=-\tfrac{1}{2}(v\zeta^{-1}-\sigma\zeta)\sqrt{-1}.$$

Hence equations (3′) give

$$r\cos(v-nt-\epsilon)=\mathrm{a}\left[1+(a_2+a_{-2})\cos 2\xi+(a_4+a_{-4})\cos 4\xi+\ldots\right],$$
$$r\sin(v-nt-\epsilon)=\mathrm{a}\left[\quad (a_2-a_{-2})\sin 2\xi+(a_4-a_{-4})\sin 4\xi+\ldots\right].$$

If it is desired to express these in polar coordinates, we have

$$v-nt-\epsilon=\tan(v-nt-\epsilon)-\tfrac{1}{3}\tan^3(v-nt-\epsilon)+\ldots.$$

As $\tan(v-nt-\epsilon)$ is a small quantity of the second order at least, the terms in the right-hand member of this equation can be calculated from the above values of $r\cos(v-nt-\epsilon)$, $r\sin(v-nt-\epsilon)$, by expansion. Also, since an equation for C has been given, the parallax $1/r$ may be found from either of the equations (20), (22) of Art. 20, after we have put z and Ω zero therein. When the numerical value of m has been used, it is simplest to make the transformations for the true longitude by the method of special values.

(ii) **The Terms whose Coefficients depend only on** m, e.

The General Solution of Equations (1).

257. We shall deduce the form of the general solution of equations (1) in the manner of Art. 247. From the results of Chaps. VI., VII., it is evident that, when all the terms whose coefficients depend only on m, e are considered, we have

$$\frac{a}{r}=\Sigma_i\Sigma_p b_{2i+pc}\cos(2i\xi+p\phi),\qquad v-n't-\epsilon'=\xi+\Sigma_i\Sigma_p b'_{2i+pc}\sin(2i\xi+p\phi),$$

where i ranges from $-\infty$ to $+\infty$ and p from 0 to ∞; the coefficients depend only on m, e and b_{2i+pc}, b'_{2i+pc} are of the order e^p at least.

Hence X, Y will be of the forms

$$\left.\begin{aligned}X&=\mathrm{a}\Sigma_i\Sigma_p A_{2i+pc}\cos\{(2i+1)\xi+p\phi\},\\ Y&=\mathrm{a}\Sigma_i\Sigma_p A'_{2i+pc}\sin\{(2i+1)\xi+p\phi\}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(9),$$

where A_{2i+pc}, A'_{2i+pc} depend only on m, e and are of the order e^p at least. If we allow p to receive negative as well as positive values, the accent of A'_{2i+pc} in Y may be omitted.

From these values of X, Y, the corresponding expressions for υ, σ may be deduced. After putting $-i-1$ for i, and $-p$ for p, in the expression for σ, we obtain

$$\left.\begin{aligned}\upsilon &= \mathrm{a}\,\Sigma_i\Sigma_p A_{2i+p\mathrm{c}} \exp.\{(2i+1)\,\xi + p\phi\}\sqrt{-1},\\ \sigma &= \mathrm{a}\,\Sigma_i\Sigma_p A_{-2i-2-p\mathrm{c}} \exp.\{(2i+1)\,\xi + p\phi\}\sqrt{-1},\end{aligned}\right\}(i, p = +\infty \ldots -\infty)\ldots(10).$$

Let $\qquad cn = \mathrm{c}\,(n-n'), \qquad \varpi - \epsilon = \mathrm{c}t_1\,(n-n')$;

so that t_1 replaces the arbitrary ϖ. We have then

$$\phi = cnt + \epsilon - \varpi = \mathrm{c}\,(n-n')\,(t-t_1), \qquad \mathrm{c} = c\,(1+\mathrm{m}).$$

If we put $\qquad \exp.\,(n-n')\,(t-t_1)\sqrt{-1} = \zeta_1,$

we have $\qquad \exp.\,\{(2i+1)\,\xi + p\phi\}\sqrt{-1} = \zeta^{2i+1}\zeta_1^{p\mathrm{c}}.$

Now in order that the values (10) may satisfy the equations (1), it is necessary to substitute them in the latter and to equate to zero the coefficients of the various powers of ζ, ζ_1^{c}. In the process of doing so, it will be necessary to use the differentials,

$$D\,(\zeta^{2i+1}\zeta_1^{p\mathrm{c}}) = (2i+1+p\mathrm{c})\,\zeta^{2i+1}\zeta_1^{p\mathrm{c}}, \qquad D^2\,(\zeta^{2i+1}\zeta_1^{p\mathrm{c}}) = (2i+1+p\mathrm{c})^2\,\zeta^{2i+1}\zeta_1^{p\mathrm{c}}.$$

Since the value of c will not be substituted in the *index* of ζ_1, if we put $\zeta_1 = \zeta$ (which corresponds to making $t_1 = t_0$) the equations of condition will be perfectly general; the only point to remember is that, when we return to real coordinates, the part of the index of ζ which contains c corresponds to the argument $\mathrm{c}\,(n-n')\,(t-t_1)$. The equations (10) may therefore be written

$$\upsilon = \mathrm{a}\,\Sigma_i\Sigma_p A_{2i+p\mathrm{c}}\,\zeta^{2i+1+p\mathrm{c}}, \qquad \sigma = \mathrm{a}\Sigma_i\Sigma_p A_{-2i-2-p\mathrm{c}}\,\zeta^{2i+1+p\mathrm{c}} \ldots(11);$$

or, in the more symmetrical forms,

$$\upsilon\zeta^{-1} = \mathrm{a}\Sigma_i\Sigma_p A_{2i+p\mathrm{c}}\zeta^{2i+p\mathrm{c}}, \qquad \sigma\zeta = \mathrm{a}\Sigma_i\Sigma_p A_{-2i-p\mathrm{c}}\,\zeta^{2i+p\mathrm{c}} \;\ldots\ldots(11').$$

Substitute the values (11) in (1). For this purpose we have, as in Art. 249,

$$D\upsilon = \mathrm{a}\Sigma_i\Sigma_p\,(2i+1+p\mathrm{c})\,A_{2i+p\mathrm{c}}\zeta^{2i+1+p\mathrm{c}}, \qquad \text{etc.};$$

$$\upsilon\sigma = \mathrm{a}^2\Sigma_j\Sigma_q\Sigma_i\Sigma_p A_{2i+p\mathrm{c}}A_{2i-2j+p\mathrm{c}-q\mathrm{c}}\,\zeta^{2j+q\mathrm{c}}, \quad \text{etc.};$$

where j, q have the same range of values as i, p, namely, from $+\infty$ to $-\infty$. Equating the coefficients of $\zeta^{2j+q\mathrm{c}}$ to zero, we obtain

$$\left.\begin{aligned}&\Sigma_i\Sigma_p\,[(2j+q\mathrm{c})^2 + (2i+p\mathrm{c}+1)\,(2i+p\mathrm{c}-2j-q\mathrm{c}+1)\\ &\qquad\qquad + 2\,(4i+2p\mathrm{c}-2j-q\mathrm{c}+2)\,\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2]\,A_{2i+p\mathrm{c}}A_{2i-2j+p\mathrm{c}-q\mathrm{c}}\\ &\qquad + \tfrac{9}{4}\mathrm{m}^2\Sigma_i\Sigma_p\,[A_{2i+p\mathrm{c}}A_{2j-2i-2+q\mathrm{c}-p\mathrm{c}} + A_{2i+p\mathrm{c}}A_{-2j-2i-2-q\mathrm{c}-p\mathrm{c}}] = 0,\\ &(2j+q\mathrm{c})\,\Sigma_i\Sigma_p\,(4i+2p\mathrm{c}-2j-q\mathrm{c}+2+2\mathrm{m})\,A_{2i+p\mathrm{c}}A_{2i-2j+p\mathrm{c}-q\mathrm{c}}\\ &\qquad - \tfrac{9}{2}\mathrm{m}^2\Sigma_i\Sigma_p\,[A_{2i+p\mathrm{c}}A_{2j-2i-2+q\mathrm{c}-p\mathrm{c}} - A_{2i+p\mathrm{c}}A_{-2j-2i-2-q\mathrm{c}-p\mathrm{c}}] = 0\end{aligned}\right\}(12),$$

except for $j = 0 = q$, when the right hand member of the first equation is C/a^2.

258. On comparing these equations with (4), we see that the latter would be the same as the former if the symbols

$$a, \qquad 2j, \qquad 2i, \qquad \Sigma_i,$$

were respectively replaced by

$$A, \quad 2j+qc, \quad 2i+pc, \quad \Sigma_i\Sigma_p.$$

In making the corresponding transformations, we can therefore use the results previously obtained. Indeed this fact was evident as soon as we had arrived at the equations (11).

To get the equations corresponding to (4′), multiply the second of equations (12) by $(2m+1)/(2j+qc)$ and subtract it from the first. The second equation is to be divided by $2(2j+qc)$. It is not necessary to write down the results, for they are immediately deducible from (4′).

A pair of corresponding coefficients will be A_{2i+pc}, A_{-2i-pc}. If, in order to isolate A_{2i+pc}, we make the transformations of Art. 251 after putting $2j+qc$, $2i+pc$ for $2j$, $2i$ respectively, we arrive at the equation

$$0=\Sigma_i\Sigma_p\{[2j+qc,\ 2i+pc]\,A_{2i+pc}A_{2i-2j+pc-qc} + [2j+qc,]\,A_{2i+pc}A_{2j-2i-2+qc-pc} + (2j+qc,)\,A_{2i+pc}A_{-2j-2i-2-qc-pc}\}\ldots(13),$$

the value $j=0=q$ being excluded. The coefficients in this equation are the values of (7′) after the changes noted have been made. We evidently have

$$[2j+qc, \quad 2j+qc]=-1, \qquad [2j+qc, \quad 0]=0;$$

so that the coefficient of the term A_0A_{2i+pc} is -1 and that of A_0A_{-2i-pc} is 0.

The equations for C and a can be obtained in a similar manner.

The equations of condition (13) are to be solved by continued approximation, but since there is a double sign of summation involved, they are by no means so simple as those of Art. 251. We know that the term A_{2i+pc} is of order $e^{|p|}$ at least; the simplest method therefore appears to consist in finding initially those terms which depend on the first power of e only, neglecting all higher powers of e, thus restricting q to the values ± 1.

If we neglect all powers of e and put $A_0=1$, the equations will reduce to (7). We have then

$$q=0, \quad p=0, \quad A_{2i}=a_{2i}, \quad A_0=a_0=1.$$

It must be remembered that when powers of e are not neglected, A_{2i} is no longer equal to a_{2i} but differs from it by terms of the order e^2 at least. The A_{2i} are the coefficients of the variational terms when powers of e are not neglected, while the a_{2i} are the coefficients of the same terms when the parts dependent on $e^2, e^4\ldots$ are neglected.

The Terms dependent on m and on the First Power of e.

259. As powers of e above the first will be neglected and as the coefficients of the variational terms contain only even powers of e, we put

$$A_{2i} = a_{2i}, \qquad A_{2i+c} = \epsilon_i, \qquad A_{2i-c} = \epsilon_i',$$

so that ϵ_i, ϵ_i' are of the form $ef(m)$.

In the equations of condition (13), q takes the values $+1$, -1 only. When $q = +1$, p has the values 1, 0 in the first two terms, and the values 0, -1 in the third term; any other values of p will give terms of the orders e^3, e^5,.... Similarly, when $q = -1$, we must give to p the values -1, 0 in the first two terms and the values 0, 1 in the third term. The equations for obtaining ϵ_j, ϵ_j' are therefore

$$0 = \Sigma_i \{[2j + c, 2i + c]\, \epsilon_i a_{2i-2j} + [2j + c, 2i]\, a_{2i} \epsilon'_{i-j}$$
$$+ [2j + c,]\, (\epsilon_i a_{2j-2i-2} + a_{2i} \epsilon_{j-i-1}) + (2j + c,)\, (a_{2i} \epsilon'_{-j-i-1} + \epsilon'_i a_{-2j-2i-2})\},$$

$$0 = \Sigma_i \{[2j - c, 2i - c]\, \epsilon'_i a_{2i-2j} + [2j - c, 2i]\, a_{2i} \epsilon_{i-j}$$
$$+ [2j - c,]\, (\epsilon'_i a_{2j-2i-2} + a_{2i} \epsilon'_{j-i-1}) + (2j - c,)\, (a_{2i} \epsilon_{-j-i-1} + \epsilon_i a_{-2j-2i-2})\}.$$

Since j receives positive and negative values, the equations of condition may be put into a more symmetrical form by considering those for ϵ_j, ϵ'_{-j} which form a 'pair.' Also, since each term is summed for all values of i, we can, in any term, put $i \pm$ integer for i. In the second term of the first equation put $-i + j$ for i; in the second part of the third term, $j - i - 1$ for i; and so on. With like changes in the second equation after $-j$ has been put for j, the equations of condition may be written,

$$\left.\begin{aligned} &\Sigma_i \{[2j + c, 2i + c]\, \epsilon_i a_{2i-2j} + [2j + c, 2j - 2i]\, \epsilon'_{-i} a_{2j-2i} \\ &\qquad + 2\, [2j + c,]\, \epsilon_i a_{2j-2i-2} + 2\, (2j + c,)\, \epsilon'_{-i} a_{2i-2j-2}\} = 0, \\ &\Sigma_i \{[-2j - c, -2i - c]\, \epsilon'_{-i} a_{2j-2i} + [-2j - c, 2i - 2j]\, \epsilon_i a_{2i-2j} \\ &\qquad + 2\, [-2j - c,]\, \epsilon'_{-i} a_{2i-2j-2} + 2\, (-2j - c,)\, \epsilon_i a_{2j-2i-2}\} = 0 \end{aligned}\right\} \quad (14).$$

260. There are no other relations satisfied by the unknown coefficients ϵ_j, ϵ'_{-j} ($j = +\infty \ldots -\infty$), and therefore the equations (14) only suffice to determine the ratios of the unknowns to one of them, say to ϵ_0 or ϵ_0'. Moreover, since the coefficients, as well as the equations (14), always occur in pairs, some relation must exist between the equations, and it is necessary to so determine the unknown constant c that the relation may be satisfied. When c has been found, the ratios of the unknowns ϵ_j, ϵ'_{-j} may be calculated by the ordinary methods of approximation. Hence one of them is an arbitrary constant, and this corresponds to the arbitrary constant which, in other theories, is denoted by e.

It is not difficult to see that the unknowns ϵ_j, ϵ'_{-j}, like the coefficients a_{2j}, are, in general, in descending order of magnitude with ascending values of $|j|$. Assuming this fact, suppose that we neglect all powers of m above the second and also the unknowns beyond those given by the values $j=\pm 1$; we shall have six equations to find the ratios of ϵ_0, ϵ_0', ϵ_{-1}, ϵ_1', ϵ_1, ϵ'_{-1}, namely, those given by $j=0$, ± 1, and the relation determining c will be expressible by a determinant with six rows. If we include the values for which $j=\pm 2$, the determinant will have 10 rows, and so on. Finally, including all the equations, the determinant will have an infinite number of rows and columns—the infinite number being of the form $4j+2$.

It is possible to approximate to c while the approximations to ϵ_j, ϵ'_{-j} are being carried out, but this would be very troublesome owing to the way in which this constant is involved in the coefficients $[2j+c, 2i]$, etc. Or it might be found by calculating the determinant after neglecting the smaller terms; this again would be very laborious owing to the large number of rows necessary to secure the accuracy required at the present day. In his paper 'The Motion of the Perigee'*, Dr Hill has shown that c may be made to depend on a symmetrical infinite determinant in which the number of rows or columns is of the form $2j+1$, and further, he has succeeded in representing the value of this determinant by means of a series whose terms diminish with great rapidity. See Arts. 262 et seq.

The homogeneous form of the equations (14) with respect to the unknowns, shows that the value of c, when powers of e above the first are neglected, is independent of the particular definition to be assigned to the eccentricity constant. The remark is made because, when finding c by de Pontécoulant's method (Art. 139), it appeared to be involved with the definition there given to e. We also notice that, when powers of e above the second are neglected, c will be a function of m only—a fact which renders the previous remark obvious from another point of view.

In working out a complete theory by this method, it will be found more convenient to put $A_{2i+c}=\epsilon_{2i}$, $A_{2i-c}=\epsilon'_{2i}$, in order that the introduction of a new symbol for the coefficients of the terms dependent on e and on powers of a/a' may be avoided. These coefficients are of the form A_{2i+1+c}, A_{2i+1-c}.

261. Suppose that c has been obtained in some way, accurately to the order ultimately required, either as a series in powers of m or numerically. The coefficients in the equations (14) can then be found and the values of the unknowns calculated.

When the numerical value of m is used at the outset, the best method of dealing with these equations is to neglect, in the first instance, the two equations given by $j=0$. The equations given by $j=\pm 1, \pm 2, \dots$ will then furnish the rest of the unknowns in terms of ϵ_0, ϵ_0', by the usual methods of approximation. Substituting these in the equations given by the value $j=0$, we shall have two equations to find the ratio ϵ_0/ϵ_0', and if the value of c, previously obtained, is correct, these should agree. A formula of verification is thus available. The arbitrary constant may be taken to be e, where

$$e=\epsilon_0-\epsilon_0'.$$

* See footnote, p. 196.

It is found that $\frac{1}{2}$e differs very little from the constant e used by Delaunay and defined in Arts. 159, 200 above.

The method outlined above will be found more fully developed in Parts I.—IV. of a paper by the author, *The Elliptic Inequalities in the Lunar Theory**. The notation used is slightly different: i and j, p and q are interchanged; for c, m are put c, m; instead of e the symbol Y_0 is used; the other differences will be obvious. The values of a_{2i}, c, as obtained by Hill, are used, and the results for ϵ_j/e, $\epsilon'_{-j}/\mathrm{e}$ are computed numerically to 10 places of decimals. The ratio of e to e is found by transforming to polar coordinates and comparing the resulting coefficient of the principal elliptic inequality in longitude with that given by purely elliptic motion.

The Determination of the Part of c *which depends only on* m.

262. The problem to be considered here is the discovery of an equation which will give c: we are not concerned with the unknown coefficients ϵ_i, ϵ_i'. Now a transformation of coordinates will not affect the periods of the various terms and we are therefore at liberty to choose those coordinates which will put the problem into the simplest form. It was noticed that, when the method of Art. 259 was followed, the determinant giving c was of infinite order and that the number of rows or columns was of the form $4j+2$. It will be shown here that c may be made to depend on an infinite determinant with only half that number of rows and columns; in other words, only half the number of rows or columns are necessary for a given degree of accuracy. When, however, the determinant has been found, instead of limiting the number of rows, we shall show how it may be expanded generally as a series, and we shall then see to what degree of approximation the *series* must be taken in order to secure a given accuracy in the results.

The results of Arts. 244–246 show that the problem may be stated as follows:—Given a periodic solution of a pair of differential equations, to find the periods of a solution differing but little from the given solution.

263. The equations (1) being equivalent to (25) of Art. 23, let the latter be written

$$\left.\begin{array}{l}\ddot{X}-2n'\dot{Y}=\partial F'/\partial X, \qquad \dot{Y}+2n'\dot{X}=\partial F'/\partial Y,\\ \dot{X}^2+\dot{Y}^2=V^2=2F'+\text{const.}=2\mu/r+3n'^2X^2+\text{const.}\end{array}\right\}\ldots\ldots(15);$$

where

F' is then independent of t explicitly, and V is the velocity with respect to the moving axes of X, Y.

Let ψ be the angle which the tangent at (X, Y) makes with the X-axis, and put

$$\frac{\partial}{\partial T}=\cos\psi\frac{\partial}{\partial X}+\sin\psi\frac{\partial}{\partial Y},\quad \frac{\partial}{\partial N}=\cos\psi\frac{\partial}{\partial Y}-\sin\psi\frac{\partial}{\partial X}\ldots\ldots(16);$$

* *Amer. Journ. Math.* Vol. xv. pp. 244–263, 321–338.

then $\partial F'/\partial T$, $\partial F'/\partial N$ represent the resolved parts of the forces corresponding to the force-function F', along the tangent and the normal respectively*.

Since $\dot{X} = V\cos\psi$, $\dot{Y} = V\sin\psi$, we deduce

$$\left.\begin{aligned}\frac{\partial^2 F'}{\partial N^2} &= \cos^2\psi\frac{\partial^2 F'}{\partial Y^2} - 2\sin\psi\cos\psi\frac{\partial^2 F'}{\partial X\partial Y} + \sin^2\psi\frac{\partial^2 F'}{\partial X^2},\\ \frac{d}{dt}\frac{\partial F'}{\partial N} &= V\frac{\partial^2 F'}{\partial N\partial T} - \frac{d\psi}{dt}\frac{\partial F'}{\partial T}\end{aligned}\right\}\ldots(16');$$

where the meanings to be attached to $\partial^2 F'/\partial N^2$, $\partial^2 F'/\partial N\partial T$ are evident.

Also, since $\tan\psi = \dot{Y}/\dot{X}$ and therefore $d\psi/dt = (\ddot{Y}\dot{X} - \ddot{X}\dot{Y})/V^2$, we obtain easily from (15), (16),

$$\frac{dV}{dt} = \frac{\partial F'}{\partial T} = \frac{1}{V}\frac{dF'}{dt}, \qquad V\left(\frac{d\psi}{dt} + 2n'\right) = \frac{\partial F'}{\partial N}\ldots\ldots\ldots\ldots(17).$$

Let δX, δY be any small variations of X, Y, which are such that $X+\delta X$, $Y+\delta Y$, as well as X, Y, satisfy the differential equations, and let δT, δN be the corresponding variations along the tangent and normal to the orbit of X, Y. Neglecting squares of δX, δY, we have

$$\delta T = \cos\psi\,\delta X + \sin\psi\,\delta Y, \qquad \delta N = \cos\psi\,\delta Y - \sin\psi\,\delta X \;\ldots\;(18);$$

and the equations (16) show that

$$\delta F' = \frac{\partial F'}{\partial T}\delta T + \frac{\partial F'}{\partial N}\delta N, \qquad \frac{\partial}{\partial N}\delta F' = \delta\frac{\partial F'}{\partial N}, \quad \text{etc.}$$

All the equations may therefore be submitted to a variation δ. We have, from (15),

$$\left.\begin{aligned}\frac{d^2}{dt^2}\delta X - 2n'\frac{d}{dt}\delta Y - n'^2\delta X &= \frac{\partial}{\partial X}\delta F' - n'^2\delta X,\\ \frac{d^2}{dt^2}\delta Y + 2n'\frac{d}{dt}\delta X - n'^2\delta Y &= \frac{\partial}{\partial Y}\delta F' - n'^2\delta Y\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(19).$$

Now the left-hand members of these equations are the accelerations of a point referred to axes moving with angular velocity n'. Take the sum of these equations multiplied by $\cos\psi$, $\sin\psi$, respectively, and also their sum multiplied by $-\sin\psi$, $\cos\psi$, respectively: the results will be the accelerations referred to the tangent and normal of the original orbit. The latter rotate with angular velocity $n' + d\psi/dt$. Since the coordinates,

* It is necessary to define the partial differentials in this way because F' is not expressible in terms of T, N only; they have the same meaning as if F' were so expressible.

referred to these axes, are δT, δN, we obtain, by the well-known formula for the acceleration in the direction of the normal*, and by (16), (18),

$$\frac{d^2}{dt^2}\delta N - \left(\frac{d\psi}{dt}+n'\right)^2\delta N + \frac{1}{\delta T}\frac{d}{dt}\left\{\left(\frac{d\psi}{dt}+n'\right)(\delta T)^2\right\} = \frac{\partial}{\partial N}\delta F' - n'^2\delta N,$$

or, $$\left\{\frac{d^2}{dt^2}+n'^2-\left(\frac{d\psi}{dt}+n'\right)^2\right\}\delta N = -\frac{d^2\psi}{dt^2}\delta T - 2\left(\frac{d\psi}{dt}+n'\right)\frac{d}{dt}\delta T + \delta\frac{\partial F'}{\partial N} \quad\text{...(20).}$$

Now

$$\delta\frac{\partial F'}{\partial N} = \frac{\partial^2 F'}{\partial N^2}\delta N + \frac{\partial^2 F'}{\partial N\partial T}\delta T = \frac{\partial^2 F'}{\partial N^2}\delta N + \frac{\delta T}{V}\left(\frac{d}{dt}\frac{\partial F'}{\partial N}+\frac{d\psi}{dt}\frac{\partial F'}{\partial T}\right), \qquad \text{by (16}'\text{)},$$

$$= \frac{\partial^2 F'}{\partial N^2}\delta N + \frac{\delta T}{V}\frac{d}{dt}\left\{V\left(\frac{d\psi}{dt}+2n'\right)\right\} + \frac{\delta T}{V}\frac{d\psi}{dt}\frac{dV}{dt}, \qquad \text{by (17)},$$

$$= \frac{\partial^2 F'}{\partial N^2}\delta N + \frac{d^2\psi}{dt^2}\delta T + 2\left(\frac{d\psi}{dt}+n'\right)\frac{dV}{dt}\frac{\delta T}{V}.$$

By means of this result, equation (20) becomes

$$\left\{\frac{d^2}{dt^2}+n'^2-\left(\frac{d\psi}{dt}+n'\right)^2-\frac{\partial^2 F'}{\partial N^2}\right\}\delta N = -2\left(\frac{d\psi}{dt}+n'\right)\left(\frac{d}{dt}\delta T - \frac{dV}{dt}\frac{\delta T}{V}\right) \quad\text{...(20}'\text{).}$$

Also, since δV is the variation of the velocity, relative to the moving axes, in the direction of the tangent, we have

$$\delta V = \frac{d}{dt}\delta T - \frac{d\psi}{dt}\delta N.$$

But, by the third of equations (15) and by (17),

$$V\delta V = \delta F' = \frac{\partial F'}{\partial N}\delta N + \frac{\partial F'}{\partial T}\delta T = \left(\frac{d\psi}{dt}+2n'\right)V\delta N + \frac{dV}{dt}\delta T.$$

Eliminating δV, we find

$$\frac{d}{dt}\delta T - \frac{dV}{dt}\frac{\delta T}{V} = 2\left(\frac{d\psi}{dt}+n'\right)\delta N.$$

The equation (20′) therefore becomes

$$\frac{d^2}{dt^2}\delta N + (n-n')^2\,\Theta\,\delta N = 0 \quad\text{......................(21),}$$

where $$(n-n')^2\,\Theta = 3\left(\frac{d\psi}{dt}+n'\right)^2 + n'^2 - \frac{\partial^2 F'}{\partial N^2} \quad\text{..............(22).}$$

This method of arriving at the equation (21) is a modification of one given by Prof. Adams in his lectures on the Lunar Theory in the academic year 1885–6. A similar investigation was given independently by Prof. G. H. Darwin in his lectures of 1893–4. The method used by Dr Hill demands several transformations†; the variable there called w is equal to $2\delta N\sqrt{-1}$.

* Tait and Steele, *Dynamics of a Particle*, Art. 42.

† *Motion of the Perigee*, pp. 6–9.

264. *Form of the Solution of the Equation for δN.*

It will be convenient to express Θ in terms of the rectangular coordinates. For this purpose we have, as before,

$$\frac{d\psi}{dt} = \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{V^2}, \quad \cos\psi = \frac{\dot{X}}{V}, \quad \sin\psi = \frac{\dot{Y}}{V}.$$

Whence, substituting for $\partial^2 F'/\partial N^2$ from (16′), we find

$$(n-n')^2\,\Theta = 3\left\{\frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{V^2} + n'\right\}^2 + n'^2 - \frac{1}{V^2}\left\{\dot{Y}^2\frac{\partial^2 F'}{\partial X^2} - 2\dot{X}\dot{Y}\frac{\partial^2 F'}{\partial X \partial Y} + \dot{X}^2\frac{\partial^2 F'}{\partial Y^2}\right\} \quad \text{.........(22′)},$$

where $$V^2 = \dot{X}^2 + \dot{Y}^2, \qquad F' = \mu\,(X^2 + Y^2)^{-\frac{1}{2}} + \tfrac{3}{2}n'^2 X^2.$$

Since the quantities present in Θ refer to the given periodic solution (which is the intermediate orbit), it is evident from equations (2′) that it is expressible by cosines of multiples of $2\xi = 2\,(n-n')\,(t-t_0)$. We can therefore express Θ in the form

$$\Theta_0 + 2\Theta_1 \cos 2\xi + 2\Theta_2 \cos 4\xi + \ldots,$$

where Θ_0, Θ_1,... are functions of m only. If $\Theta_{-i} = \Theta_i$, equation (21) becomes

$$\frac{d^2}{dt^2}\delta N + (n-n')^2\left(\sum_{i=-\infty}^{\infty}\Theta_i \cos 2i\xi\right)\delta N = 0 \quad \text{..............(23)}.$$

This equation is of the form referred to in Art. 146, and it admits of a solution of a similar nature. In order to retain the connection with the previous methods, the form of the solution will be deduced from our previous knowledge of the forms of δX, δY.

We have, from (18),

$$\delta N = \cos\psi\,\delta Y - \sin\psi\,\delta X = (\dot{X}\,\delta Y - \dot{Y}\,\delta X)/V.$$

Since $\dot{X}$, $\dot{Y}$ are respectively expressible by means of sines and cosines of odd multiples of ξ, and δY, δX, from (9), by sines and cosines of the angles $(2i+1)\,\xi \pm \phi$, the function $\dot{X}\delta Y - \dot{Y}\delta X$ will be expressible by means of cosines of the angles $2i\xi \pm \phi$; also V and $1/V$ are expressible by means of cosines of the angles $2i\xi$. Hence δN will be expressible by means of cosines of angles of the form

$$2i\,(n-n')\,(t-t_0) \pm \mathrm{c}\,(n-n')\,(t-t_1)$$
$$= (2i \pm \mathrm{c})\,(n-n')\,(t-t_0) \pm \mathrm{c}\,(n-n')\,(t_0-t_1).$$

Introducing the operator D, defined as before by means of the relation $D^2 = -\,d^2/(n-n')^2\,dt^2$, equation (23) becomes

$$D^2\delta N = (\Sigma_i \Theta_i \zeta^{2i})\,\delta N \quad \text{........................(23′)};$$

where $i = +\infty \ldots -\infty$, and $\Theta_{-i} = \Theta_i$. The general solution, according to the remarks just made, will be of the form

$$\delta N = (\Sigma_i \mathrm{b}_i \zeta^{2i+\mathrm{c}}) \exp. \mathrm{c}\,(n-n')\,(t_0-t_1)\sqrt{-1} + (\Sigma_i \mathrm{b}_i' \zeta^{2i-\mathrm{c}}) \exp. -\mathrm{c}\,(n-n')\,(t_0-t_1)\sqrt{-1}.$$

On substituting this expression in (23′) and equating the coefficients of the several powers of ζ to zero, we see that, since i receives negative as well as positive values, the equations of condition for the coefficients b_i are the same as those for the coefficients b'_{-i}. It is therefore only necessary for the determination of c to consider the integral

$$\delta N = \Sigma_i \mathrm{b}_i \zeta^{2i+\mathrm{c}} \qquad (24).$$

This result, which has been deduced from our previous knowledge of the form of the solution, is a well known property of equations of the form (23′). All that now remains is the substitution of (24) in (23′), and the deduction of c from the equations of condition obtained by equating the coefficients of the various powers of ζ to zero.

265. In order to bring Θ into a form suitable for calculation, it will be better to express (22) in terms of the complex variables v, σ. We have

$$V^2 = -(n-n')^2 Dv D\sigma,$$

and

$$V^2 \frac{d\psi}{dt} = \ddot{Y}\dot{X} - \ddot{X}\dot{Y} = -\tfrac{1}{2}(n-n')^3 (D^2 v D\sigma - D^2\sigma Dv), \qquad \frac{\partial}{\partial N} = \frac{\sqrt{-1}}{(Dv D\sigma)^{\frac{1}{2}}}\left(Dv\frac{\partial}{\partial v} - D\sigma\frac{\partial}{\partial \sigma}\right);$$

therefore, from (22),

$$\Theta = 3\left\{\tfrac{1}{2}\left(\frac{D^2 v}{Dv} - \frac{D^2\sigma}{D\sigma}\right) + \mathrm{m}\right\}^2 + \mathrm{m}^2 + \tfrac{1}{2}\frac{Dv}{D\sigma}\frac{\partial^2 \Omega_0}{\partial v^2} - \frac{\partial^2\Omega_0}{\partial v \partial\sigma} + \tfrac{1}{2}\frac{D\sigma}{Dv}\frac{\partial^2\Omega_0}{\partial\sigma^2},$$

where $\quad \Omega_0 = 2F''/(n-n')^2 = 2\kappa\,(v\sigma)^{-\frac{1}{2}} + \tfrac{3}{4}\mathrm{m}^2 (v+\sigma)^2$:

the transformation used in Arts. 18, 19.

But, from Art. 18,

$$D^2 v + 2\mathrm{m} Dv = -\frac{\partial\Omega_0}{\partial\sigma}, \qquad D^2\sigma - 2\mathrm{m}D\sigma = -\frac{\partial\Omega_0}{\partial v};$$

therefore $$\frac{D^2 v}{Dv} + \frac{D^2\sigma}{D\sigma} = -\frac{1}{D\sigma}\frac{\partial\Omega_0}{\partial v} - \frac{1}{Dv}\frac{\partial\Omega_0}{\partial\sigma}.$$

Hence

$$-\tfrac{1}{2}D\left(\frac{D^2 v}{Dv} + \frac{D^2\sigma}{D\sigma}\right) = \tfrac{1}{2}\frac{Dv}{D\sigma}\frac{\partial^2\Omega_0}{\partial v^2} + \tfrac{1}{2}\frac{D\sigma}{Dv}\frac{\partial^2\Omega_0}{\partial\sigma^2} + \frac{\partial^2\Omega_0}{\partial v\partial\sigma} - \tfrac{1}{2}\frac{D^2\sigma}{(D\sigma)^2}\frac{\partial\Omega_0}{\partial v} - \tfrac{1}{2}\frac{D^2 v}{(Dv)^2}\frac{\partial\Omega_0}{\partial\sigma}.$$

The last two terms, by the equations of motion, are equal to

$$\tfrac{1}{2}\left(\frac{D^2 v}{Dv}\right)^2 + \tfrac{1}{2}\left(\frac{D^2\sigma}{D\sigma}\right)^2 + \mathrm{m}\left(\frac{D^2 v}{Dv} - \frac{D^2\sigma}{D\sigma}\right) = \left\{\tfrac{1}{2}\left(\frac{D^2 v}{Dv} - \frac{D^2\sigma}{D\sigma}\right) + \mathrm{m}\right\}^2 + \tfrac{1}{4}\left(\frac{D^2 v}{Dv} + \frac{D^2\sigma}{D\sigma}\right)^2 - \mathrm{m}^2,$$

and, since $r^2 = v\sigma$, $$\frac{\partial^2\Omega_0}{\partial v\partial\sigma} = \tfrac{1}{2}\frac{\kappa}{r^3} + \tfrac{3}{2}\mathrm{m}^2;$$

hence
$$\tfrac{1}{2}\frac{D\sigma}{Dv}\frac{\partial^2\Omega_0}{\partial\sigma^2}+\tfrac{1}{2}\frac{Dv}{D\sigma}\frac{\partial^2\Omega_0}{\partial v^2}-\frac{\partial^2\Omega_0}{\partial v\partial\sigma}=-\tfrac{1}{2}D\left(\frac{D^2v}{Dv}+\frac{D^2\sigma}{D\sigma}\right)-\left(\frac{\kappa}{r^3}+2\mathrm{m}^2\right)$$
$$-\left\{\tfrac{1}{2}\left(\frac{D^2v}{Dv}-\frac{D^2\sigma}{D\sigma}\right)+\mathrm{m}\right\}^2-\tfrac{1}{4}\left(\frac{D^2v}{Dv}+\frac{D^2\sigma}{D\sigma}\right)^2,$$
and therefore
$$\Theta=-\left(\frac{\kappa}{r^3}+\mathrm{m}^2\right)+2\left\{\tfrac{1}{2}\left(\frac{D^2v}{Dv}-\frac{D^2\sigma}{D\sigma}\right)+\mathrm{m}\right\}^2-\left\{\tfrac{1}{2}\left(\frac{D^2v}{Dv}+\frac{D^2\sigma}{D\sigma}\right)\right\}^2-D\left\{\tfrac{1}{2}\left(\frac{D^2v}{Dv}+\frac{D^2\sigma}{D\sigma}\right)\right\}.$$

The quantities present in this equation have to be expanded in powers of ζ. Assume
$$D^2v/Dv=\Sigma_i U_i\zeta^{2i},\qquad \kappa/r^3+\mathrm{m}^2=2\Sigma_i M_i\zeta^{2i}\ \dots\dots\dots\dots\dots\dots\dots\dots(25);$$
then
$$D^2\sigma/D\sigma=-\Sigma_i U_i\zeta^{-2i},\qquad M_{-i}=M_i.$$

When U_i, M_i have been found, the calculation of Θ requires only two operations of any length, namely, the squaring of two series.

We have, after the substitution of the values (3) of v, σ,
$$\Sigma_i(2i+1)^2a_{2i}\zeta^{2i}=[\Sigma_i(2i+1)a_{2i}\zeta^{2i}][\Sigma_i U_i\zeta^{2i}];$$
from which, by equating coefficients, the coefficients U_i can be obtained.

The coefficients M_i can be found by treating, in the same way, the equation
$$D^2v+2\mathrm{m}Dv+\tfrac{3}{2}\mathrm{m}^2\sigma+\tfrac{5}{2}\mathrm{m}^2v=v(\mathrm{m}^2+\kappa/r^3)=2v\Sigma_i M_i\zeta^{2i}.$$

When the numerical value of m is used, it is easier to calculate the coefficients U_i, M by the method of special values, after eliminating the second differentials by means of the equations of motion. It is evident that Θ_i will be of the order $\mathrm{m}^{|2i|}$ at least.

The Equation for c.

266. When the value (24) is substituted in (23′), we have, on equating the coefficient of ζ^{2j+c} to zero,
$$(\mathrm{c}+2j)^2\mathrm{b}_j-\Sigma_i\Theta_{j-i}\mathrm{b}_i=0,\qquad (i=+\infty\ \dots\ -\infty);$$
or, since $\Theta_{-i}=\Theta_i$,
$$\dots-\Theta_2\mathrm{b}_{j-2}-\Theta_1\mathrm{b}_{j-1}+\{(\mathrm{c}+2j)^2-\Theta_0\}\mathrm{b}_j-\Theta_1\mathrm{b}_{j+1}-\Theta_2\mathrm{b}_{j+2}-\dots=0;$$
by giving to j the values $0, \pm 1, \pm 2, \dots$, we obtain a set of linear homogeneous equations. Since the unknowns and the equations are infinite in number, some remarks concerning the treatment of them are necessary.

Suppose that the two series of quantities $\mathrm{b}_0, \mathrm{b}_{\pm1}, \mathrm{b}_{\pm2}, \dots, \Theta_0, \Theta_1, \Theta_2, \dots$ are in descending order of magnitude, and let all the coefficients beyond b_p, b_{-p}, Θ_p (p a positive integer) be neglected; we shall have $2p+1$ unknowns and $2p+1$ equations. As the equations are linear and homogeneous with respect to the unknowns, a relation, which may be expressed by equating to zero a determinant of $2p+1$ rows, must exist between the equations. We shall assume that the same results hold when p becomes infinitely great (see the note at the end of Art. 267).

We thus suppose that the ordinary rules for treating a set of linear homogeneous equations may be used when the unknowns and the equations are infinite in number. To every positive value of j there corresponds an equal negative value, and there is one equation for $j=0$: we may therefore consider that the number of equations is odd and that the coefficient of b_0, in the equation given by $j=0$, is the central constituent of the determinant formed by eliminating the unknowns. Let the two equations, obtained by equating to zero the coefficients of $\zeta^{\pm 2j+c}$, be divided by $4j^2-\Theta_0$. The condition that the new series of equations, thus formed, shall be consistent is expressed by the equation

$$\Delta(c)=0,$$

where $\Delta(c)$ represents the infinite symmetrical determinant,

$$\left|\begin{array}{cccccc}
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\ldots, & \dfrac{(c-4)^2-\Theta_0}{4^2-\Theta_0}, & -\dfrac{\Theta_1}{4^2-\Theta_0}, & -\dfrac{\Theta_2}{4^2-\Theta_0}, & -\dfrac{\Theta_3}{4^2-\Theta_0}, & -\dfrac{\Theta_4}{4^2-\Theta_0}, \ldots \\
\ldots, & -\dfrac{\Theta_1}{2^2-\Theta_0}, & \dfrac{(c-2)^2-\Theta_0}{2^2-\Theta_0}, & -\dfrac{\Theta_1}{2^2-\Theta_0}, & -\dfrac{\Theta_2}{2^2-\Theta_0}, & -\dfrac{\Theta_3}{2^2-\Theta_0}, \ldots \\
\ldots, & -\dfrac{\Theta_2}{0^2-\Theta_0}, & -\dfrac{\Theta_1}{0^2-\Theta_0}, & \dfrac{c^2-\Theta_0}{0^2-\Theta_0}, & -\dfrac{\Theta_1}{0^2-\Theta_0}, & -\dfrac{\Theta_2}{0^2-\Theta_0}, \ldots \\
\ldots, & -\dfrac{\Theta_3}{2^2-\Theta_0}, & -\dfrac{\Theta_2}{2^2-\Theta_0}, & -\dfrac{\Theta_1}{2^2-\Theta_0}, & \dfrac{(c+2)^2-\Theta_0}{2^2-\Theta_0}, & -\dfrac{\Theta_1}{2^2-\Theta_0}, \ldots \\
\ldots, & -\dfrac{\Theta_4}{4^2-\Theta_0}, & -\dfrac{\Theta_3}{4^2-\Theta_0}, & -\dfrac{\Theta_2}{4^2-\Theta_0}, & -\dfrac{\Theta_1}{4^2-\Theta_0}, & \dfrac{(c+4)^2-\Theta_0}{4^2-\Theta_0}, \ldots \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}\right|$$

267. The equation $\Delta(c)=0$ may be regarded as an equation for c with an infinite number of roots which have the following properties:—

(i) *The roots occur in pairs.* For when $-c$ is put for c, $\Delta(c)$ remains unaltered. Hence, if c_0 be a root of the equation, $-c_0$ is another root.

(ii) *If c_0 be a root, the expressions*

$$c=\pm c_0+2j \qquad (j=-\infty\ldots+\infty)$$

are also roots. Let $c+2$ be put for c, and let the divisors $4j^2-\Theta_0$ (which are infinitely great at infinity) be all moved on one place; the central line and column are merely altered in position, and the determinant therefore remains unaltered. The same result holds if c be increased or diminished by any even integer.

Since these roots are also roots of the equation $\cos\pi c=\cos\pi c_0$, we have

$$\Delta(c)=k(\cos\pi c-\cos\pi c_0) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26).$$

(iii) *All the roots of* $\Delta(c) = 0$ *are included in the expression* $\pm c_0 + 2j$, that is, k *is independent of* c. Since $\Theta_0, \Theta_1, \Theta_2, \ldots$ do not contain c, the number of roots cannot be altered by giving any special values to $\Theta_1, \Theta_2, \ldots$; let them be put zero, and let the resulting value of $\Delta(c)$ be denoted by $\Delta_0(c)$. We have evidently

$$\Delta_0(c) = \prod_{j=-\infty}^{+\infty} \frac{(c+2j)^2 - \Theta_0}{4j^2 - \Theta_0} \quad \text{.....................} \quad (27).$$

The roots of the equation $\Delta_0(c) = 0$ are therefore the same as those of the equation

$$\cos \pi c - \cos \pi \sqrt{\Theta_0} = 0;$$

whence it follows that the roots of $\Delta(c) = 0$ are the same as those of the equation

$$\cos \pi c - \cos \pi c_0 = 0.$$

This result, combined with (26), shows that k is independent of c.

(iv) *Any root* c_0 *satisfies the equation*

$$\sin^2 \tfrac{1}{2}\pi c_0 = \Delta(0) \sin^2 \tfrac{1}{2}\pi \sqrt{\Theta_0} \quad \text{....................} \quad (28),$$

where $\Delta(0)$ *is the determinant obtained by putting* $c = 0$ *in* $\Delta(c)$.

Since (26) is true for all values of c, the value of k may be found by equating the coefficients of the highest power of c in the two members of the equation. But the highest power of c, contained in $\Delta(c)$, is obtained from the elements of the leading diagonal only; hence k is independent of $\Theta_1, \Theta_2, \ldots$. Let zeros be put for the latter quantities; $\Delta(c)$ then reduces to $\Delta_0(c)$ and c_0 to $\sqrt{\Theta_0}$. We therefore have

$$\Delta_0(c) = k(\cos \pi c - \cos \pi \sqrt{\Theta_0}).$$

Putting $c = 0$ in this identity, we obtain

$$k(1 - \cos \pi \sqrt{\Theta_0}) = \Delta_0(0) = 1, \qquad \text{by (27).}$$

Finally, substituting this value of k and putting $c = 0$ in (26), we deduce the required equation (28).

Since the substitution of $\pm c_0 + 2j$ for c_0 leaves (28) unaltered, this equation gives the periods of *all* the terms in the solution. The period we require is known to be that of the principal term, and therefore, in obtaining c_0 from the value of $\sin^2 \frac{1}{2}\pi c_0$, we must choose that value nearest to unity.

The only step that now remains, for the determination of c_0, is the calculation of $\Delta(0)$. Every element of the leading diagonal of this determinant is unity and the other elements are functions of m which may be

found according to the methods explained in Art. 266. Putting $c = 0$ in $\Delta(c)$, we find that $\Delta(0)$ is equal to

$$\left|\begin{array}{cccccccc} \multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ \dots, & 1, & -\frac{\Theta_1}{4^2-\Theta_0}, & -\frac{\Theta_2}{4^2-\Theta_0}, & -\frac{\Theta_3}{4^2-\Theta_0}, & -\frac{\Theta_4}{4^2-\Theta_0}, & \dots \\ \dots, & -\frac{\Theta_1}{2^2-\Theta_0}, & 1, & -\frac{\Theta_1}{2^2-\Theta_0}, & -\frac{\Theta_2}{2^2-\Theta_0}, & -\frac{\Theta_3}{2^2-\Theta_0}, & \dots \\ \dots, & -\frac{\Theta_2}{0^2-\Theta_0}, & -\frac{\Theta_1}{0^2-\Theta_0}, & 1, & -\frac{\Theta_1}{0^2-\Theta_0}, & -\frac{\Theta_2}{0^2-\Theta_0}, & \dots \\ \dots, & -\frac{\Theta_3}{2^2-\Theta_0}, & -\frac{\Theta_2}{2^2-\Theta_0}, & -\frac{\Theta_1}{2^2-\Theta_0}, & 1, & -\frac{\Theta_1}{2^2-\Theta_0}, & \dots \\ \dots, & -\frac{\Theta_4}{4^2-\Theta_0}, & -\frac{\Theta_3}{4^2-\Theta_0}, & -\frac{\Theta_2}{4^2-\Theta_0}, & -\frac{\Theta_1}{4^2-\Theta_0}, & 1, & \dots \\ \multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}\right|$$

The complete examination of the assumptions involved in the preceding results is outside the limits of this treatise; they have been considered by Poincaré*. We shall only give below the conditions which must be fulfilled in order that an infinite determinant may be expanded according to the ordinary methods.

268. *Convergency of an Infinite Determinant.*

Let a determinant of $2n+1$ rows or columns be denoted by Δ_n; the determinant Δ_n is said to be convergent if it continually approaches a finite and determinate limit as n approaches to infinity.

Let the element in the ith row above the middle row and in the jth column to the right of the middle column be denoted by $\beta_{i,j}$, and the element of the principal diagonal in the ith row by $1+\beta_{i,-i}$, where $i, j = +n \dots -n$; in other words, let $\beta_{i,j}$ be a non-diagonal element and $1+\beta_{i,-i}$ a diagonal element, the central element being $1+\beta_{0,0}$. Consider the continued product†

$$\Pi_i\left(|1+\beta_{i,-i}| + \Sigma_j |\beta_{i,j}|\right),$$

where, under the sign of summation, the value $j = -i$ is excluded. This expression, which we shall for brevity denote by Π_n, contains all the terms of the development of Δ_n and other terms besides. Since, by definition, all the terms of Π_n have positive signs, Δ_n is less than Π_n and therefore, when n is infinite,

$$\text{Lim. } \Delta_n < \text{Lim. } \Pi_n.$$

But
$$\Pi_n \leqq \Pi_i\left(1+\Sigma_j |\beta_{i,j}|\right),$$

where the value $j = -i$ is not excluded from the second member. Whence, when n becomes infinite,

$$\text{Lim. } \Delta_n < \text{Lim. } \Pi_i\left(1+\Sigma_i |\beta_{i,j}|\right).$$

The second expression is convergent if $\Sigma_{i,j} |\beta_{i,j}|$ converges‡. This condition includes the convergence of $\Sigma_i |\beta_{i,-i}|$ which is the condition of convergence of the product $\Pi_i(1+\beta_{i,-i})$.

* Sur les déterminants d'ordre infini. *Bull. de la Soc. Math. de France*, Vol. XIV. pp. 77–90.

† See footnote, p. 201.

‡ See G. Chrystal, *Algebra*, Pt. II. p. 137. E. W. Hobson, *Trigonometry*, Arts. 279–281.

Hence the limit of Δ_n is finite if *the sum of the non-diagonal elements be absolutely convergent and the product of the elements in the principal diagonal be absolutely convergent.*

Again, the expansion of the determinant Δ_n can be deduced from that of Δ_{n+p} (p a positive integer), by putting zeros for certain of the elements which are present in the latter but not present in the former. Hence $\Delta_{n+p}-\Delta_n$ represents the terms which vanish when these elements are annulled. But $\Pi_{n+p}-\Pi_n$ must contain all these terms (and possibly others) all affected with a positive sign, and Π_{n+p} contains every term present in Π_n. Hence

$$\Delta_{n+p}-\Delta_n \leqq \Pi_{n+p}-\Pi_n.$$

But, if $\Sigma_{i,j}|\beta_{i,j}|$ be convergent, $\Pi_{n+p}-\Pi_n$ and therefore $\Delta_{n+p}-\Delta_n$ can be made less than any finite quantity by sufficiently increasing n. That is, the conditions given in italics are sufficient for the convergence of Δ_n when n is infinite*.

The condition sufficient for the convergence of the determinant $\Delta(0)$ can be easily deduced. The second of the conditions, given above, is evidently satisfied, for all the elements in the principal diagonal are unity. The first condition is satisfied if

$$(\Sigma_i|\Theta_i|)\left(\Sigma_j\left|\frac{1}{4j^2-\Theta_0}\right|\right)$$

be convergent. The latter series is well known to be convergent. Hence, in order that $\Delta(0)$ may be convergent, it suffices that $\Sigma_i|\Theta_i|$ be convergent. We shall assume that this condition is satisfied.

The Development of an Infinite Determinant.

269. It will be convenient to denote the non-diagonal element $\beta_{i,j}$ by $(i:j)$, and the diagonal element $1+\beta_{i,-i}$ by $(i:-i)$†. The central constituent is then $(0:0)$, and the term in the development arising from the product of the elements of the leading diagonal is

$$\ldots(i:-i)(i-1:-i+1)\ldots(1:-1)(0:0)(-1:1)\ldots\ldots(-i:i)\ldots.$$

Any other term in the development may be obtained from this by interchanging any of the second numbers of the several symbols contained in the above expression, the first numbers remaining unchanged‡.

If one change be made between two of the second numbers, the other elements remaining the same, the term is said to be produced by *one exchange*; for example, if we interchange $-i$, 0, so that the elements $(i:-i)(0:0)$ become $(i:0)(0:-i)$, the new term is said to be produced by one exchange.

Two exchanges may be made in two ways. They may either be made amongst three elements (e.g. we may exchange $-i$, 0 and then 0, -1), or they may be made amongst four elements (e.g. by exchanging $-i$, 0 and also 1, i). Similarly *three exchanges* may be made

* The proof, when the elements of the principal diagonal are all unity, was given by Poincaré in the paper just referred to. The proof given here is from a paper by H. von Koch, 'Sur les Déterminants Infinis, &c.,' *Acta. Math.*, Vol. XVI. pp. 217–295.

† The explanation will be more easily followed by taking the principal row and principal column as axes: the positive direction of the y-axis being to the right and that of the x-axis upwards. The 'coordinates' of any element are then i, j.

‡ Of course the development may be also made by interchanging the first numbers, the second numbers remaining unchanged.

amongst four, five, or six elements. Any term produced by n exchanges which might, by a different proceeding, have been produced by $n-k$ exchanges, is excluded from the consideration of the terms produced by n exchanges, since it is considered in the $n-k$ exchanges. Further, it is evident that the terms produced by n exchanges have a positive or negative sign according as n is even or odd.

In the following, i is an integer which may have any value, including zero, between $-\infty$ and $+\infty$; $k, k', k'',\ldots$ are integers having any values from 1 to $+\infty$.

270. Let all the elements of the leading diagonal be unity. The term in the development obtained by using all the elements of the leading diagonal is therefore 1.

Consider any two elements of the leading diagonal

$$(i : -i), \quad (i+k : -i-k).$$

Owing to the fact that k can only be a positive integer greater than zero, these expressions will always denote different elements, and their product will never denote the same pair of elements for different pairs of values of i, k.

One exchange between these gives

$$(i : -i-k), \quad (i+k : -i).$$

Since all the other elements of the leading diagonal are unity, the expression

$$-\Sigma_i\Sigma_k(i : -i-k)(i+k : -i), \qquad (i=-\infty\ldots+\infty,\ k=1\ldots\infty)\ldots\ldots(29),$$

gives the terms, in the development of the determinant, obtained by one exchange.

Consider three elements of the leading diagonal

$$(i : -i), \quad (i+k : -i-k), \quad (i+k+k' : -i-k-k').$$

There are only two possible ways of making two exchanges amongst these three elements, so that none of them remain in the leading diagonal. They are

$$(i : -i-k), \quad (i+k : -i-k-k'), \quad (i+k+k' : -i)$$

and

$$(i : -i-k-k'), \quad (i+k : -i), \quad (i+k+k' : -i-k).$$

Hence, all the terms arising from two exchanges amongst *three* elements are

$$+\Sigma_i\Sigma_k\Sigma_{k'}\{(i : -i-k)(i+k : -i-k-k')(i+k+k' : -i) \\ +(i : -i-k-k')(i+k : -i)(i+k+k' : -i-k)\}\ldots\ldots(29').$$

Two exchanges amongst the *four* elements

$$(i : -i), \quad (i+k : -i-k), \quad (i+k+k' : -i-k-k'), \quad (i+k+k'+k'' : -i-k-k'-k''),$$

so that none of them remain in the leading diagonal, are given by exchanging $-i$, $-i-k$ and exchanging $-i-k-k'$, $-i-k-k'-k''$; the combinations obtained by exchanging $-i$, $-i-k-k'$ or $-i$, $-i-k-k'-k''$ are included. Hence the terms in the development, obtained by making two exchanges amongst *four* elements, are

$$\Sigma_i\Sigma_k\Sigma_{k'}\Sigma_{k''}(i : -i-k)(i+k : -i)(i+k+k' : -i-k-k'-k'')(i+k+k'+k'' : -i-k-k')\ldots(29'').$$

In a similar manner, we may consider the terms of the development, produced by three or more exchanges.

Development of $\Delta(0)$.

271. Let us now apply these results to the development of $\Delta(0)$. We have

$$(i: -i-k) = -\frac{\Theta_k}{4i^2-\Theta_0} = -\frac{\Theta_{-k}}{4i^2-\Theta_0} = (-i: i+k).$$

Let $4i^2-\Theta_0=a_i$, so that $a_i=a_{-i}$.

Now $\Theta_k=\Theta_{-k}$ is of the order m^{2k} at least; it is therefore evident that the order of any element $(i:j)$ is $2|i+j|$. Since k, k' are always positive and greater than zero, we see immediately that the terms (29), produced by one exchange, will be of the fourth order at least; the terms (29′), (29″), produced by two exchanges, will be of the eighth order at least; similarly, the terms produced by n exchanges will be of the order $4n$ at least. We shall here neglect terms of the 12th and higher orders.

The terms (29) give

$$-\Sigma_i\Sigma_k \frac{\Theta_k^2}{a_i a_{i+k}} = -\Sigma_i\left(\frac{\Theta_1^2}{a_i a_{i+1}}+\frac{\Theta_2^2}{a_i a_{i+2}}\right), \qquad \text{to } \mathrm{m}^{11}.$$

Since $\Theta_{-k}=\Theta_k$, etc., the terms obtained from (29′) are

$$-2\Sigma_i\Sigma_k\Sigma_{k'}\frac{\Theta_k\Theta_{k'}\Theta_{k+k'}}{a_i a_{i+k}a_{i+k+k'}} = -2\Sigma_i\frac{\Theta_1^2\Theta_2}{a_i a_{i+1}a_{i+2}}, \qquad \text{to } \mathrm{m}^{11};$$

since values of k, k' greater than 1 give terms of an order higher than the eleventh.

The terms (29″) give

$$+\Sigma_i\Sigma_k\Sigma_{k'}\Sigma_{k''}\frac{\Theta_k^2\Theta_{k''}^2}{a_i a_{i+k}a_{i+k+k'}a_{i+k+k'+k''}} = \Theta_1^4\Sigma_i\Sigma_{k'}\frac{1}{a_i a_{i+1}a_{i+k'+1}a_{i+k'+2}}, \qquad \text{to } \mathrm{m}^{11}.$$

Hence, as far as the eleventh order inclusive, we have

$$\Delta(0) = 1-\Sigma_i\left\{\frac{\Theta_1^2}{a_i a_{i+1}}+\frac{\Theta_2^2}{a_i a_{i+2}}+\frac{2\Theta_1^2\Theta_2}{a_i a_{i+1}a_{i+2}}\right\}+\Sigma_i\Sigma_{k'}\frac{\Theta_1^4}{a_i a_{i+1}a_{i+k'+1}a_{i+k'+2}} \quad\ldots\ldots(30).$$

272. The final process consists in replacing the summations, in the expression (30), by finite terms. For this purpose we have, if k be a finite integer*,

$$\pi\cot\pi a = \Sigma_i\frac{1}{a+i} = \Sigma_i\frac{1}{a-i} = \Sigma_i\frac{1}{a+i+k} = \Sigma_i\frac{1}{a-i-k}, \qquad (i=\infty\ldots-\infty).$$

Here a is never equal to an integer, and, in the method used below, the semi-convergency of these forms will not affect the values of the functions which they are supposed to represent.

Let $\Theta_0=4a^2$; then $a_i=4i^2-\Theta_0=4(i^2-a^2)$. Hence, decomposing into partial fractions,

$$\Sigma_i\frac{16}{a_i a_{i+k}} = \Sigma_i\frac{1}{[a^2-i^2][a^2-(i+k)^2]} = \Sigma_i\left[\frac{A}{a+i}+\frac{B}{a-i}+\frac{C}{a+i+k}+\frac{D}{a-i-k}\right],$$

where $\quad 1/A=1/D=2ka(2a-k), \quad 1/B=1/C=-2ka(2a+k).$

Therefore, summing each of the four terms by means of the formulæ given above,

$$\Sigma_i\frac{1}{a_i a_{i+k}} = \frac{\pi\cot\pi a}{16ka}\left(\frac{1}{2a-k}-\frac{1}{2a+k}\right) = \frac{\pi\cot\pi a}{8a(4a^2-k^2)} \quad\ldots\ldots\ldots\ldots(31).$$

By giving to k the values 1, 2, successively, the second and third terms of (30) may be obtained from this result.

* E. W. Hobson, *Trigonometry*, Art. 293.

In the same manner, $1/a_i a_{i+k} a_{i+k'}$ may be decomposed into partial fractions and summed for all values of i; then, by giving to k, k' the values 1, 2, respectively, the fourth term of (30) may be found.

The last term may be obtained by decomposing $1/a_i a_{i+1} a_{i+k'+1} a_{i+k'+2}$ into partial fractions, as before, and summing for all values of i. The result is

$$\frac{1}{64}\,\frac{\{20a^2-(k'+1)^2-1\}\,\pi\cot\pi a}{a\,(4a^2-1)\,\{4a^2-(k'+1)^2\}\,\{4a^2-(k'+2)^2\}\,(4a^2-k'^2)}.$$

Replacing $2a$ by its value, this may be again decomposed and summed for all values of k' from 1 to ∞. The decomposition is best effected by putting k for $k'+1$; the expression can then be exhibited as a function of k^2. After summation for all values of k from 0 to ∞, the terms for $k=0$, $k=1$ must be subtracted.

The results are given by Dr Hill so as to include all terms of an order less than the sixteenth*. As far as the sixth (since $1-\Theta_0$ is of the first) order, we have found in equations (30), (31),

$$\Delta(0)=1+\tfrac{1}{4}\pi\,\{\Theta_1{}^2\,(1-\Theta_0)^{-1}\,\Theta_0{}^{-\frac{1}{2}}\}\cot\tfrac{1}{2}\pi\sqrt{\Theta_0}.$$

Taking that value of c_0, obtained from (28), which is nearest to unity, this expression for $\Delta(0)$ gives $c_0=1{\cdot}07158\ 28$. The value obtained by Dr Hill† to 15 places of decimals, with $m={\cdot}08084\ 89338\ 08312$, is $c_0=1{\cdot}07158\ 32774\ 16012$, giving

$$1-c_0=1-c_0/(1+m)={\cdot}00857\ 25730\ 04864.$$

He has also obtained c_0 literally, to the order m^{11}, from the equations of condition‡.

The Terms whose Coefficients depend on m and on the Second and Higher Powers of e.

273. *The Terms of Order e^2.* There are two classes of terms to be considered, namely, those terms whose characteristics are zero and those whose characteristics are e^2. The former are the parts of the Variational Inequalities, that is, of the terms with arguments $2i\xi$, whose coefficients are of the order e^2; the latter are those Elliptic Inequalities whose coefficients are of the order e^2 and whose arguments are of the form $2i\xi\pm 2\phi$. We shall only indicate the method of procedure.

(a) *The Parts of the Variational Inequalities which are of the Order e^2.*

For these it is necessary to put $q=0$ in equation (13), but, instead of neglecting in the results all terms which depend on e—a proceeding which gave the equations (7)—we must give to p such values that terms of order e^2 are included. In the first two terms p therefore takes the values 0, ± 1, and in the third term the values ± 1, 0. The equation obtained from the coefficient of ζ_1^{2j} becomes, on combining those terms multiplied by $[2j,]$ $(2j,)$ and containing the suffix c, which are equal when $j-i-1$ is put for i, etc.,

$$\begin{aligned}0=\Sigma_i\{&[2j,\,2i]\,A_{2i}A_{2i-2j}+[2j,\,2i+c]\,A_{2i+c}A_{2i-2j+c}+[2j,\,2i-c]\,A_{2i-c}A_{2i-2j-c}\\&+[2j,]\,(A_{2i}A_{2j-2i-2}+2A_{2i+c}A_{2j-2i-2-c})+(2j,)\,(A_{2i}A_{-2j-2i-2}+2A_{2i+c}A_{-2j-2i-2-c})\},\end{aligned}$$

except for $j=0$. It will be noticed that those terms whose factors involve c, are at least

* *Motion of the Perigee*, p. 32.

† *Id.* p. 35.

‡ G. W. Hill, "Literal Expression for the Motion of the Moon's Perigee," *Annals of Math.* (U. S. A.), Vol. IX. pp. 31–41.

of the second order with respect to e; hence the known part of c, which depends only on m and which is denoted above by c_0, is sufficiently accurate.

Let $$A_{2i} = a_{2i} + \delta a_{2i},$$

where a_{2i} is the coefficient found in Section (i) and δa_{2i} is the new part of the order e^2. Also, neglecting powers of e above the second, we have

$$A_{2i+c} = \epsilon_i, \quad A_{2i-c} = \epsilon_i'.$$

Substituting these in the equation just given and making use of equations (7), we obtain, to the order required,

$$\Sigma_i \{[2j, 2i](a_{2i}\delta a_{2i-2j} + a_{2i-2j}\delta a_{2i}) + [2j, 2i+c]\,\epsilon_i\epsilon_{i-j} + [2j, 2i-c]\,\epsilon'_i\epsilon'_{i-j}$$
$$+ 2[2j,](a_{2j-2i-2}\delta a_{2i} + \epsilon_i\epsilon'_{j-i-1}) + 2(2j,)(a_{-2j-2i-2}\delta a_{2i} + \epsilon_i\epsilon'_{-j-i-1})\} = 0.$$

The quantities a_{2i}, ϵ_i, ϵ_i' having been already found, we have a series of linear equations for the determination of the coefficients δa_i; but since the value $j=0$ is excluded, these equations are only sufficient to determine $\delta a_{\pm 1}$, $\delta a_{\pm 2}, \ldots$ in terms of m, e^2, δa_0. It is necessary to see how δa_0 may be found.

Owing to the introduction of a in the values of v, σ, one of the members of the product aA_0 is arbitrary. When the inequalities dependent on m only were considered, we put $A_0 = a_0 = 1$ (Art. 258); since we are now considering the terms dependent on e^2 as well as those dependent on m, the definition of A_0 must be extended so as to cover these terms. It is found to be most convenient, in making the calculations after the method outlined here, to define A_0 in the same way as before, so that

$$\delta a_0 = 0.$$

The same definition holds when the terms of the orders e^4, e'^2, etc. are under consideration; the new part of a, of the order e^2, can then be found by an extension of the method of Art. 255. The equations, which are linear with constant terms, are now sufficient to determine $\delta a_{\pm 1}$, $\delta a_{\pm 2}, \ldots$ in terms of e^2, m, and the method of continued approximation may be used to find the unknowns.

It may be remarked that, when the method, referred to at the end of Art. 287 below, is used, it is more convenient to define a to be such, that its value obtained in Art. 255 is unaltered by any of the subsequent approximations; thus δa$=0$, but δa_0 is no longer zero. In this method, the requisite number of equations for finding δa_0, $\delta a_{\pm 1}, \ldots$ appear naturally.

(*b*) *The Terms whose Characteristics are* e^2.

These terms have arguments of the form $2i\xi \pm 2\phi$. We therefore put $q = +2$, $q = -2$, successively, in the equation (13), such values being given to p that terms of a higher order shall not occur. All the terms are directly seen to be at least of the order e^2. We then obtain, for the determination of the unknown coefficients, a series of linear equations which correspond in number to the unknowns, and which contain constant terms; all the unknowns can therefore be found by continued approximation. Since all the terms are of the order e^2, the value of c to be substituted in the coefficients $[2j+2c, 2i]$, etc. is known with sufficient accuracy.

274. *The Terms of Order* e^3. These again are of two kinds: those with arguments $2i\xi \pm \phi$, $2i\xi \pm 3\phi$. For the latter we put $q = \pm 3$ in the equations (13); all the terms present are of order e^3, and the determination of the unknowns proceeds on exactly the same lines as that of the terms whose characteristics are e^2. The terms of arguments

$2i\xi \pm \phi$ are troublesome because the value of c, already found, is not sufficiently accurate: it must be known to the order e^2.

Put $q = +1, -1$ successively in equations (13) and give to p such values that terms of the order e^3 may be included. The only coefficients present will be found to be A_{2i}, A_{2i+c}, A_{2i-c}, A_{2i+2c}, A_{2i-2c}. Let

$$A_{2i} = a_{2i} + \delta a_{2i}, \quad A_{2i+c} = \epsilon_i + \delta\epsilon_i, \quad A_{2i-c} = \epsilon_i' + \delta\epsilon_i', \quad c = c_0 + \delta c;$$

$\delta\epsilon_i$, $\delta\epsilon_i'$ are then of order e^3 and δa_{2i}, δc of order e^2. The terms which are of the order e vanish by equations (14); the coefficients A_{2i+2c}, A_{2i-2c} are known by Art. 273 (b). We have remaining a set of linear equations for the determination of δc, $\delta\epsilon_i$, $\delta\epsilon_i'$, containing known terms independent of these quantities.

Since one of the coefficients $A_{2i\pm c}$ was arbitrary, the same must be true of one of the $\delta\epsilon_i$, $\delta\epsilon_i'$; this fact is also deducible from the consideration that the number of unknowns is greater by one than the number of equations. It is most convenient to determine the arbitrary so that, when *all* powers of e are included,

$$A_c - A_{-c} = e = \epsilon_0 - \epsilon_0';$$

so that $\delta\epsilon_0 = \delta\epsilon_0'$. With this assumption the values of $\delta\epsilon_i$, $\delta\epsilon_i'$, δc may be obtained by continued approximation.

The method of carrying out the approximations outlined in this and in the previous article may be found in Pts. v.—ix. of the paper referred to in Art. 261 above. The results are obtained numerically as far as m is concerned, the value of e being left arbitrary.

The fact that one of the $\delta\epsilon_i$, $\delta\epsilon_i'$ is arbitrary implies that some relation free from these quantities and containing only δc can be obtained from the equations of condition. The author has given a definite form to this relation, with several extensions, on pp. 336—338 of a paper entitled *Investigations in the Lunar Theory**. The method of obtaining it will be outlined in connection with the latitude inequalities and the determination of δg. See Arts. 284, 285, 288 below.

(iii) **The Terms whose Coefficients depend only on** m, e'.

275. Since the terms dependent on the solar parallax and on the latitude are neglected, we put $\Omega = \Omega_2$, $z = 0$ in the equations (23), (19) of Art. 20. Hence, it is necessary to add to the right-hand members of the equations (1) of Art. 245 the terms

$$-3\Omega_2 + D^{-1}(D_t\Omega_2), \qquad \sigma\frac{\partial\Omega_2}{\partial\sigma} - \upsilon\frac{\partial\Omega_2}{\partial\upsilon},$$

respectively. But when z is neglected, Ω_2 is of the form (Art. 128),

$$\Omega_2 = A\upsilon^2 + 2B\upsilon\sigma + C\sigma^2,$$

where A, B, C depend only on the coordinates of the Sun. The terms to be added to the right-hand members of equations (1) are therefore

$$-3(A\upsilon^2 + 2B\upsilon\sigma + C\sigma^2) + D^{-1}(\upsilon^2DA + 2\upsilon\sigma DB + \sigma^2DC), \quad 2C\sigma^2 - 2A\upsilon^2 \ldots (32).$$

* *Amer. Journ. Math.* Vol. xvii. pp. 318–358.

To find A, B, C we have, by Art. 19,

$$\Omega_2 = 3\mathrm{m}^2 \left\{\frac{a'^3}{r'^3} r^2 S^2 - \tfrac{1}{4}(\upsilon + \sigma)^2\right\} - \mathrm{m}^2 \upsilon\sigma \left(\frac{a'^3}{r'^3} - 1\right).$$

But, by Art. 22,

$$\begin{aligned} rS &= (XX' + YY')/r' \\ &= \tfrac{1}{2}(\upsilon + \sigma)\cos(v' - n't - \epsilon') - \tfrac{1}{2}\sqrt{-1}\,(\upsilon - \sigma)\sin(v' - n't - \epsilon'). \end{aligned}$$

Let $v' - n't - \epsilon' = V'$. Then

$$rS = \tfrac{1}{2}\upsilon \exp.(-V'\sqrt{-1}) + \tfrac{1}{2}\sigma \exp.(+V'\sqrt{-1}),$$

$$r^2 S^2 = \tfrac{1}{4}\upsilon^2 \exp.(-2V'\sqrt{-1}) + \tfrac{1}{4}\sigma^2 \exp.(+2V'\sqrt{-1}) + \tfrac{1}{2}\upsilon\sigma.$$

Substituting in Ω_2, we find

$$\mathrm{A} = \tfrac{3}{4}\mathrm{m}^2 \left\{\frac{a'^3}{r'^3}\exp.(-2V'\sqrt{-1}) - 1\right\}, \qquad \mathrm{C} = \tfrac{3}{4}\mathrm{m}^2 \left\{\frac{a'^3}{r'^3}\exp.(2V'\sqrt{-1}) - 1\right\},$$

$$\mathrm{B} = \tfrac{1}{4}\mathrm{m}^2 \left(\frac{a'^3}{r'^3} - 1\right);$$

whence $\quad \mathrm{C} + \mathrm{A} = \tfrac{3}{2}\mathrm{m}^2\left(\dfrac{a'^3}{r'^3}\cos 2V' - 1\right), \quad \mathrm{C} - \mathrm{A} = \tfrac{3}{2}\mathrm{m}^2\sqrt{-1}\,\dfrac{a'^3}{r'^3}\sin 2V'.$

Since $V' = v' - \varpi' - (n't + \epsilon' - \varpi') = f' - w'$ (Arts. 48, 53), the only functions which have to be calculated are

$$a'^3/r'^3, \qquad (a'^3/r'^3)\cos 2(f' - w'), \qquad (a'^3/r'^3)\sin 2(f' - w').$$

These are expanded in powers of e' and in cosines and sines of multiples of w' after the manner explained in Arts. 39—41. The results for them have been fully worked out by several investigators*.

Assume $$\frac{a'^3}{r'^3} - 1 = \Sigma_p \alpha_p' \cos pw',$$

$$\frac{a'^3}{r'^3}\cos 2V' - 1 = \Sigma_p \beta_p' \cos pw', \qquad \frac{a'^3}{r'^3}\sin 2V' = \Sigma_p \beta_p' \sin pw',$$

where $p = -\infty \ldots +\infty$ and $\alpha'_{-p} = \alpha_p'$; then α_p', β_p' are known functions of e'. Using these expressions, we have

$$\mathrm{A} = \tfrac{3}{4}\mathrm{m}^2 \Sigma_p \beta_p' e^{-pw'\sqrt{-1}}, \quad \mathrm{C} = \tfrac{3}{4}\mathrm{m}^2 \Sigma_p \beta_p' e^{pw'\sqrt{-1}}, \quad \mathrm{B} = \tfrac{1}{4}\mathrm{m}^2 \Sigma_p \alpha_p' e^{pw'\sqrt{-1}}.$$

Since the coefficients of t in ζ^{m} and $e^{w'\sqrt{-1}}$ are the same, we can put ζ^{m} for $e^{w'\sqrt{-1}}$ if we remember that, when returning to real variables, $e^{pw'\sqrt{-1}}$ is to be put for $\zeta^{p\mathrm{m}}$; the value of m will not be substituted in the index of ζ. Hence we can write

$$\mathrm{A} = \tfrac{3}{4}\mathrm{m}^2 \Sigma_p \beta'_{-p} \zeta^{p\mathrm{m}}, \quad \mathrm{C} = \tfrac{3}{4}\mathrm{m}^2 \Sigma_p \beta_p' \zeta^{p\mathrm{m}}, \quad \mathrm{B} = \tfrac{1}{4}\mathrm{m}^2 \Sigma_p \alpha_p' \zeta^{p\mathrm{m}}.$$

* See the references given in Arts. 123, 126.

These values must be substituted in the terms (32), and the results added to the right-hand members of equations (1).

276. It is evident that the required solution of the equations is a particular integral corresponding to the newly added terms. Since e is neglected, the solution will be of the form

$$v = \mathrm{a}A_{2i+p\mathrm{m}}\,\zeta^{2i+1+p\mathrm{m}}, \qquad \sigma = \mathrm{a}A_{-2i-2-p\mathrm{m}}\,\zeta^{2i+1+p\mathrm{m}}.$$

These values being substituted in the equations, and the coefficients of $\zeta^{2j+q\mathrm{m}}$ equated to zero, we shall have a series of equations of condition to determine the unknown quantities.

The method is similar to that used in Case (ii). Neglecting, initially, powers of e' higher than the first, the values $v = v_0$, $\sigma = \sigma_0$, given by equations (3), can be used in the right-hand members, since A, B, C are at least of the order e'. We obtain a series of equations similar to (14); but since their right-hand members are no longer zero, there is no relation like that which was necessary to find c: this is otherwise evident, for the index of ζ in all these terms is quite known and no new arbitrary constant is required. The higher approximations proceed in the same way. The inverse operation D^{-1} will introduce divisors of the form $2j + q\mathrm{m}$; but as m is assumed to be incommensurable with any integer, none of these divisors will vanish.

Some results, obtained by the author for inequalities dependent on e', will be found in two Notes in the *Mon. Not. R. A. S.* Vols. LIV. p. 471, LV. pp. 3—5.

(iv) The Terms whose Coefficients depend only on m, $1/a'$.

277. Since e' is neglected we have

$$\Omega_2 = 0, \qquad D_t\Omega = 0;$$

therefore the terms to be added to the right-hand members of equations (1) are respectively

$$-\sum_{p=3}^{\infty}(p+1)\,\Omega_p, \qquad \sigma\frac{\partial\Omega}{\partial\sigma} - v\frac{\partial\Omega}{\partial v} \quad \ldots\ldots\ldots\ldots\ldots \quad (33).$$

Also, by Arts. 19, 22, we have $rS = X = \frac{1}{2}(v+\sigma)$, $r' = a'$, and therefore, as z is neglected,

$$\Omega = \frac{\mathrm{m}^2}{a'}\left\{\tfrac{5}{8}(v+\sigma)^3 - \tfrac{3}{2}v\sigma(v+\sigma)\right\} + \frac{\mathrm{m}^2}{a'^2}\left\{\tfrac{35}{64}(v+\sigma)^4 - \tfrac{15}{8}v\sigma(v+\sigma)^2 + \tfrac{3}{4}v^2\sigma^2\right\} + \ldots$$

$$= \frac{\mathrm{m}^2}{a'}\left\{\tfrac{5}{8}(v^3+\sigma^3) + \tfrac{3}{8}v\sigma(v+\sigma)\right\} + \frac{\mathrm{m}^2}{a'^2}\left\{\tfrac{35}{64}(v^4+\sigma^4) + \tfrac{5}{16}v\sigma(v^2+\sigma^2) + \tfrac{9}{32}v^2\sigma^2\right\} + \ldots.$$

The terms to be added to the equations are therefore of the third, fourth,... degrees in v, σ, corresponding to terms of the first, second,... degrees with respect to $1/a'$.

If, in the expressions (33), the values υ_0, σ_0, which are odd power series in ζ and which correspond to the intermediate orbit, be substituted for υ, σ, the terms produced by Ω_p will be odd or even power series in ζ according as p is odd or even. The values of υ, σ, when terms dependent on the solar parallax are included, will therefore contain even as well as odd powers of ζ. Hence we assume

$$\upsilon = \mathrm{a}\Sigma_i A_{i-1}\zeta^i, \qquad \sigma = \mathrm{a}\Sigma_i A_{-i-1}\zeta^i.$$

This solution includes the intermediate orbit. When $1/a'$ is neglected, we have $A_{2i} = a_{2i}$ and $A_{2i-1} = 0$.

The process is similar to that of Case (iii). First, neglecting $1/a'^2$ and higher powers, we find the odd coefficients A_{2i-1}; the parts of A_{2i} which are of the order $1/a'^2$ are next found; and so on. Since the equations of condition at any stage of the process are linear, with known terms in their right-hand members, no relation, independent of these coefficients, exists and the unknowns can be calculated by continued approximation.

For the details of the calculations, two papers by the writer—'On the Part of the Parallactic Inequalities in the Moon's Motion which is a Function of the Mean Motions of the Sun and Moon*,' and 'On the Determination of a Certain Class of Inequalities in the Moon's Motion†'—may be consulted. See also G. W. Hill, 'The Periodic Solution as a first approximation in the Lunar Theory‡.'

(v) The Terms whose Coefficients depend only on m, γ.

278. When e', $1/a'$ are neglected, $\Omega = 0$, and the equations (23), (19), (18) of Chap. II. become

$$\left.\begin{aligned} D^2(\upsilon\sigma + z^2) - D\upsilon D\sigma - (Dz)^2 - 2\mathrm{m}(\upsilon D\sigma - \sigma D\upsilon) + \tfrac{9}{4}\mathrm{m}^2(\upsilon+\sigma)^2 - 3\mathrm{m}^2 z^2 = C, \\ D(\upsilon D\sigma - \sigma D\upsilon - 2\mathrm{m}\upsilon\sigma) + \tfrac{3}{2}\mathrm{m}^2(\upsilon^2 - \sigma^2) = 0 \end{aligned}\right\} (34);$$

$$D^2 z - z(\mathrm{m}^2 + \kappa/r^3) = 0 \qquad\qquad (35).$$

The form of the last equation shows that its integral contains a constant factor which is one arbitrary of the solution. In the case of the Moon this factor is small: it has been denoted in previous chapters by γ. If, in the first two equations, terms of the order γ^2 be neglected, they reduce to those of Art. 244; the new parts of υ, σ are therefore at least of the order γ^2. We shall neglect the constant of eccentricity and consider the first approximation to the solution of equations (34), (35) to be the intermediary.

The procedure is the same as in the previous cases. We suppose the intermediate orbit known and consider first the terms of the order γ^1; these

* *Amer. Journ. Math.* Vol. XIV. pp. 141–160.
† *Mon. Not. R. A. S.* Vol. LII. pp. 71–80.
‡ *Astron. Journ.* Vol. XV. pp. 137–143.

only occur in z and they are obtained from equations (35) by inserting therein that value of r which corresponds to the intermediary. The terms whose coefficients are of order γ^2 only occur in v, σ and they are found, when those of order γ^1 have been obtained, from the equations (34); the terms of the order γ^3 only occur in z and they are obtained from (35); and so on.

The solution may be represented by

$$v = v_0 + v_{\gamma^2} + \ldots, \quad \sigma = \sigma_0 + \sigma_{\gamma^2} + \ldots, \quad z = z_\gamma + z_{\gamma^3} + \ldots;$$

where $v_{\gamma 2i}$, $\sigma_{\gamma 2i}$, $z_{\gamma 2i+1}$ represent the terms whose *coefficients* are of the orders denoted by the suffixes. Therefore, neglecting powers of γ above the third and remembering that $r^2 = v\sigma + z^2$, $r_0^2 = v_0\sigma_0$, where r_0, v_0, σ_0 refer to the intermediary, we have

$$\frac{1}{r^3} = \frac{1}{\{(v_0 + v_{\gamma^2})(\sigma_0 + \sigma_{\gamma^2}) + z_\gamma^2\}^{\frac{3}{2}}} = \frac{1}{r_0^3}\left(1 - \tfrac{3}{2}\,\frac{v_0\sigma_{\gamma^2} + \sigma_0 v_{\gamma^2} + z_\gamma^2}{r_0^2}\right).$$

Substituting, equation (35) becomes

$$D^2(z_\gamma + z_{\gamma^3}) - \left(\mathrm{m}^2 + \frac{\kappa}{r_0^3}\right)(z_\gamma + z_{\gamma^3}) = -\tfrac{3}{2}\,\frac{\kappa z_\gamma}{r_0^5}(v_0\sigma_{\gamma^2} + \sigma_0 v_{\gamma^2} + z_\gamma^2)\ldots(36):$$

the differential equation for the terms in z as far as the order γ^3.

Instead of (35), we may use an equation free from the divisor r^3. This equation—deduced immediately from equations (17), (18) of Art. 19, after putting $\Omega = 0$—is

$$D(zDv - vDz) + 2\mathrm{m}zDv + \tfrac{3}{2}\mathrm{m}^2 z(v + \sigma) + \mathrm{m}^2 zv = 0$$

(or a similar equation in which σ, v, $-D$ replace v, σ, D). This new form will be found useful for a literal development in powers of m; the method given in the text is less laborious when the numerical value of m is used at the outset.

The Terms dependent on m *and on the First Power of* γ. *Principal Part of the Motion of the Node.*

279. When terms above the order γ^1 are neglected, the equation (36) reduces to

$$D^2 z_\gamma - \left(\mathrm{m}^2 + \frac{\kappa}{r_0^3}\right) z_\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (37).$$

Substituting for $\mathrm{m}^2 + \kappa/r_0^3$ the series $2\Sigma_i M_i \zeta^{2i}$ (Art. 265), in which $M_{-i} = M_i$, $i = -\infty \ldots +\infty$, we obtain

$$D^2 z_\gamma - 2z_\gamma \Sigma_i M_i \zeta^{2i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (37').$$

Since M_i is of the order $\mathrm{m}^{|2i|}$ at least, this equation is of exactly the same form as (23′) and it may be treated in an exactly similar manner. The two independent integrals are given by

$$z_\gamma = \Sigma_j K_j \zeta^{2j+g}, \qquad z_\gamma = \Sigma_j K'_{-j}\zeta^{-2j-g} \qquad (j = -\infty \ldots +\infty),$$

and the substitution of either of these will give a series of linear homogeneous equations, from which the ratios of the unknown coefficients may be found; the infinite determinant, formed by eliminating the unknowns, will give g.

280. In order to connect this solution with that obtained in Art. 147, let the latter be written

$$z = 2\mathrm{a}\Sigma K_j \sin(2j\xi + \eta), \qquad (j = -\infty \ldots +\infty).$$

As with the elliptic inequalities (Art. 257) we put

$$\zeta^{2j} = \exp. 2\xi\sqrt{-1} = \exp. 2(n - n')(t - t_0)\sqrt{-1},$$

$$\zeta_2^{\mathrm{g}} = \exp. \eta\sqrt{-1} = \exp. (gnt + \epsilon - \theta)\sqrt{-1} = \exp. \mathrm{g}(n - n')(t - t_2)\sqrt{-1};$$

so that $\quad \mathrm{g} = gn/(n - n') = g(1 + \mathrm{m}), \qquad \mathrm{g}t_2(n - n') = \theta - \epsilon.$

If it be recollected that t_0 is to be replaced by t_2 in the part of the index of ζ which contains g, the solution may be written in the form

$$z_\gamma\sqrt{-1} = \mathrm{a}\Sigma_j(K_j\,\zeta^{2j+\mathrm{g}} + K'_{-j}\,\zeta^{-2j-\mathrm{g}}) \ldots\ldots\ldots\ldots\ldots\ldots (38),$$

where $K'_{-j} = -K_j$.

When this is substituted in equation (37′) and the coefficient of $\zeta^{2j+\mathrm{g}}$ equated to zero, we have

$$(2j + \mathrm{g})^2 K_j - 2\Sigma_i M_{j-i} K_i = 0, \qquad (i, j = -\infty \ldots +\infty) \ldots (39).$$

The equations for K'_{-j}, obtained by equating to zero the coefficients of $\zeta^{-2j-\mathrm{g}}$, are of the same form. The elimination of the K_j or of the K'_{-j} gives an infinite determinant $\nabla(\mathrm{g})$. This determinant being of the same form as $\Delta(\mathrm{c})$, all the results proved in Arts. 266—272 will be available if we replace Θ_i by $2M_i$ and c by g.

The part of g which depends only on m, may therefore be found by taking the value of g_0, nearest to unity, given by the equation

$$\sin^2 \tfrac{1}{2}\pi \mathrm{g}_0 = \nabla(0) \sin^2 \tfrac{1}{2}\pi\sqrt{2M_0};$$

where $\nabla(0)$ denotes the determinant $\Delta(0)$ of Art. 267, after Θ_i has been replaced by $2M_i$.

The determination of g_0 by this method was first made by Adams*. With the value $m = n'/n = \cdot0748013$ (exactly) or $\mathrm{m} = \cdot08084\ 89030\ 51852$, he finds $\mathrm{g} = 1\cdot08517\ 13927\ 46869$, giving $g - 1 = \cdot00399\ 91618\ 46592$. Mr P. H. Cowell has verified this value and he has obtained the literal and numerical values of g_0, K_j, as well as those of the terms of orders γ^2, γ^3.

* "On the Motion of the Moon's Node in the case when the Orbits of the Sun and Moon are supposed to have no Eccentricities, and when their Mutual Inclination is supposed to be small," *Mon. Not. R. A. S.* Vol. xxxviii. pp. 43–49; *Coll. Works*, pp. 181–188.

281. When g_0 has been found, the coefficients K_j can be determined in terms of one of them, say of K_0, by means of equations (39). We first leave aside the equation given by $j=0$; when the other coefficients have been found in terms of K_0, their values should satisfy this equation: it is therefore useful for purposes of verification.

The coefficient of $\sin\eta$, that is, of $(\zeta^{\mathrm{g}}-\zeta^{-\mathrm{g}})/2\sqrt{-1}$ will be $2\mathrm{a}K_0$, since $K_0'=-K_0$. In Art. 147 this was denoted by $a\gamma$. Hence

$$a\gamma = 2\mathrm{a}K_0.$$

The ratio $\mathrm{a}:a$ being previously known, we have the new constant of latitude in terms of the one used in Chap. VII. The relation will be modified when terms of the order γ^3 are considered.

The Terms of Order γ^2.

282. It is not necessary to give detailed explanations concerning the calculation of these terms. They are deduced from equations (34), when z_γ has been found, in exactly the same manner as the terms in e^2 were deduced when those of order e had been found. We give to z the value z_γ just obtained and put $v=v_0+v_{\gamma^2}$, $\sigma=\sigma_0+\sigma_{\gamma^2}$, rejecting, after the substitution, all terms of an order higher than γ^2. The solution divides into the parts which depend on ζ^{2i} and $\zeta^{2i\pm2\mathrm{g}}$, respectively, and the coefficients are calculated in the manner explained in Art. 273. The value g_0 of g is sufficiently approximate.

We suppose then that v, σ or v_{γ^2}, σ_{γ^2} are known correctly to the order γ^2.

The Terms of Order γ^3.

The Part of the Motion of the Node which is of Order γ^2.

283. Returning to the equation (36), we substitute in its right-hand member the values of z_γ, v_{γ^2}, σ_{γ^2}, v_0, σ_0, r_0, previously obtained, since this portion is of the order γ^3 at least. Also, from its symmetry with respect to v, σ, we see that it will be expressible in terms of $(\zeta^{2i+\mathrm{g}}-\zeta^{-2i-\mathrm{g}})$, $(\zeta^{2i+3\mathrm{g}}-\zeta^{-2i-3\mathrm{g}})$, with known coefficients. With regard to the left-hand member, $\mathrm{m}^2+\kappa/r_0^3$ is expressible as before in terms of $\zeta^{2i}+\zeta^{-2i}$. We cannot, however, put $D^2z_\gamma-(\mathrm{m}^2+\kappa/r_0^3)\,z_\gamma=0$, for, when the value of z_γ is substituted, D^2z_γ contains g in its coefficients. Denoting the value of g, to the order γ^2, by $\mathrm{g}_0+\delta\mathrm{g}$, we see that D^2z_γ will produce terms of the order $\gamma\delta\mathrm{g}$, that is, of the order γ^3.

Assume as the solution

$$z\sqrt{-1}=(z_\gamma+z_{\gamma^3})\sqrt{-1}=\mathrm{a}\,\Sigma_j\left\{(K_j+\delta K_j)\,\zeta^{2j+\mathrm{g}}+(K'_{-j}+\delta K'_{-j})\,\zeta^{-2j-\mathrm{g}}\right\}$$
$$+\text{ terms dependent on }\zeta^{2j\pm3\mathrm{g}},$$

where $K'_{-j} = -K_j$ and $\delta K'_{-j} = -\delta K_j$. The terms of the second line will be all of the order γ^3, since the index of ζ contains $\pm 3g$: they can be equated to the terms of corresponding form on the right-hand side of (36), and the unknown coefficients found in the usual way. The value g_0 of g is evidently sufficient for these terms. We shall therefore leave them aside and consider only the equations of condition obtained by equating the powers of $\zeta^{2j\pm g}$ to zero.

284. With this understanding we have, since $g = g_0 + \delta g$,

$$D^2 z\sqrt{-1} = a\,\Sigma_j (2j + g_0 + \delta g)^2 \{(K_j + \delta K_j)\,\zeta^{2j+g} + (K'_{-j} + \delta K'_{-j})\,\zeta^{-2j-g}\},$$

$$= a\,\Sigma_j (2j + g_0)^2 (K_j\,\zeta^{2j+g} + K'_{-j}\,\zeta^{-2j-g})$$

$$+ 2a\,\delta g\,\Sigma_j (2j + g_0)(K_j\,\zeta^{2j+g} + K'_{-j}\,\zeta^{-2j-g}) + a\Sigma_j (2j + g_0)^2(\delta K_j\zeta^{2j+g} + \delta K'_{-j}\zeta^{-2j-g}),$$

omitting terms of an order higher than γ^3.

When the latter expression is substituted in (36), the terms under the first sign of summation cancel with $-(m^2 + \kappa/r_0^3)\,z_\gamma$, by (37′); the remaining terms are all of order γ^3. We have therefore

$$2a\delta g\,\Sigma_j (2j + g_0)(K_j\,\zeta^{2j+g} + K'_{-j}\,\zeta^{-2j-g}) + a\Sigma_j (2j + g_0)^2 (\delta K_j\,\zeta^{2j+g} + \delta K'_{-j}\,\zeta^{-2j-g})$$

$$- a\,(m^2 + \kappa/r_0^3)\,\Sigma_j (\delta K_j\,\zeta^{2j+g} + \delta K'_{-j}\,\zeta^{-2j-g}) = -\tfrac{3}{2}\frac{\kappa z_\gamma}{r_0^5}(\upsilon_0\sigma_{\gamma^2} + \sigma_0\upsilon_{\gamma^2} + z_\gamma^2)\sqrt{-1},$$

those terms on the right, depending on $\zeta^{2j\pm 3g}$, being left aside. Equating to zero the coefficient of ζ^{2j+g} and putting $m^2 + \kappa/r_0^3 = 2\Sigma_i M_i \zeta^{2i}$, we deduce

$$2\delta g\,(2j + g_0)\,K_j + (2j + g_0)^2\,\delta K_j - 2\Sigma_i M_{j-i}\,\delta K_i$$

$$= coef.\ of\ \zeta^{2j+g}\ in\ -\tfrac{3}{2}\frac{\kappa z_\gamma\sqrt{-1}}{a r_0^5}(\upsilon_0\sigma_{\gamma^2} + \sigma_0\upsilon_{\gamma^2} + z_\gamma^2)\ldots(40).$$

The number of equations obtained from this by giving to j the values $0, \pm 1, \pm 2, \ldots$ is one less than the number of unknowns δK_j, δg; but since K_0 was arbitrary, δK_0 must be also arbitrary and it may be determined at will: when the arbitrary value, to be given to δK_0, has been fixed, the number of unknowns will correspond to the number of equations. As δg is of the order γ^2, while δK_0 is of the order γ^3, the value of δg is independent of the value to be given to δK_0, and therefore some relation, independent of the unknowns δK_j but involving δg, must exist between these equations. We shall now find this relation.

285. *The Equation for* δg.

We have

$$-\tfrac{3}{2} z_\gamma (\upsilon_0\sigma_{\gamma^2} + \sigma_0\upsilon_{\gamma^2} + z_\gamma^2)/r_0^5 = terms\ of\ order\ \gamma^3\ in\ the\ expansion$$

$$of\ z_\gamma\,\{(\upsilon_0 + \upsilon_{\gamma^2})(\sigma_0 + \sigma_{\gamma^2}) + z_\gamma^2\}^{-\frac{3}{2}}\ in\ powers\ of\ \gamma.$$

Let $$R^2_{\gamma^2} = (v_0 + v_{\gamma^2})(\sigma_0 + \sigma_{\gamma^2}) + z_\gamma^2 = \textit{value of } r^2 \textit{ correct to } \gamma^2.$$

Substituting, we see that the right-hand member of (40) is

The part, of order γ^3, *of the coef. of* ζ^{2j+g} *in the expansion of* $\kappa z_\gamma \sqrt{-1}/aR^3_{\gamma^2}$.

Multiply (40) by K_j and sum for all values of j. Since $M_{i-j} = M_{j-i}$, the terms involving the unknowns δK_j on the left-hand side will be

$$\Sigma_j \Sigma_i \{(2j + g_0)^2 K_j \delta K_j - 2M_{i-j} K_j \delta K_i\}.$$

As j, i have the same range of values, we may interchange them in the second term of this expression which then becomes

$$\Sigma_j \{(2j + g_0)^2 K_j - 2\Sigma_i M_{j-i} K_i\} \delta K_j.$$

This is zero because the coefficient of δK_j vanishes for all values of j by equations (39). Hence all the unknowns δK_j disappear.

The result of the summation of the equations (40) is therefore

$$2\delta g \Sigma_j (2j + g_0) K_j^2 = \Sigma_j [K_j \times \textit{coef., order } \gamma^3, \textit{ of } \zeta^{2j+g} \textit{ in } \kappa z_\gamma / aR^3_{\gamma^2}] \sqrt{-1}.$$

In an exactly similar manner we may find

$$2\delta g \Sigma_j (2j + g_0) K'^2_{-j} = \Sigma_j [K'_{-j} \times \textit{coef., order } \gamma^3, \textit{ of } \zeta^{-2j-g} \textit{ in } \kappa z_\gamma / aR^3_{\gamma^2}] \sqrt{-1}.$$

But since $K'_{-j} = -K_j$, equation (38) may be written

$$-z_\gamma \sqrt{-1} = a\,\Sigma_j (K'_{-j} \zeta^{2j+g} + K_j \zeta^{-2j-g}),$$

and therefore aK_j, aK'_{-j} are respectively the coefficients of ζ^{-2j-g}, ζ^{2j+g} in $-z_\gamma \sqrt{-1}$. Whence, adding the two previous equations and putting $K'^2_{-j} = K_j^2$, we have

$$4\delta g\, \Sigma_j (2j + g_0) K_j^2 = \Sigma_j [(\textit{coef. of } \zeta^{-2j-g} \textit{ in } z_\gamma)(\textit{coef., order } \gamma^3, \textit{ of } \zeta^{2j+g} \textit{ in } \kappa z_\gamma / a^2 R^3_{\gamma^2})$$
$$+ (\ldots\ldots\ldots\ldots \zeta^{2j+g} \ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \zeta^{-2j-g} \ldots\ldots\ldots\ldots)],$$

or, $$4\delta g\, \Sigma_j (2j + g_0) K_j^2 = \textit{Part, order } \gamma^4, \textit{ of const. term in } \kappa z_\gamma^2 / a^2 R^3_{\gamma^2} \quad \ldots(41);$$

where $\kappa z_\gamma^2 / a^2 R^3_{\gamma^2}$ is supposed to be expanded in powers of γ and in cosines of multiples of the arguments contained therein. Since all the other quantities present in this equation have been previously found, it constitutes a simple equation for finding δg.

286. We have seen that δK_0 may be determined at will. It may be fixed so that the coefficient of $\sin \eta$ in z is $2aK_0$ at any stage of the approximations and therefore in the final results; hence $\delta K_0 = 0$. When δg has been found, the equations obtained by putting $j = \pm 1, \pm 2, \ldots$ in (40) will enable us to find the coefficients $\delta K_{\pm 1}$, $\delta K_{\pm 2}, \ldots$ by continued approximation. The equation given by $j = 0$ is then superfluous: it may be used as an equation of verification for the values of δg, δK_j, when these have been calculated.

(vi) Terms dependent on m and on Combinations of e', e, γ, $1/a'$ and their Powers.

287. The developments which have been given above suffice to indicate the general method by which all the solar inequalities in the motion of the Moon may be found. The only real difficulties which occur are those arising when the motions of the Perigee and of the Node are required. The infinite determinant gives the principal parts of these with an accuracy which leaves nothing to be desired; the other smaller portions can all be found by simple equations in the manner explained in Art. 285.

The coefficients of the terms of any order in e, e', γ, $1/a'$ will always be determinable by a set of linear equations when the terms of lower orders in these constants have been found. It will be noticed from what has gone before that, at every stage, *all* powers of m are included, and also that, when a term of argument $\pm i'\xi \pm p\phi \pm q\phi' \pm j\eta$ is being found, all the terms of arguments $(2i \pm i')\xi \pm p\phi \pm q\phi' \pm j\eta$ ($i' = 0$ or 1 and i, p, q, j, positive integers or zeros) are determined at the same time. The process is therefore reduced to the approximate solution of a certain number of sets of linear equations—the number of such sets depending on the order in e, e', γ, $1/a'$ to which it is desired to take the solution.

After finding the intermediate orbit and the principal parts of the motions of the perigee and of the node, the use of the equations (19), (23) of Art. 20 for obtaining the other inequalities is not essential. We may return to one of the original equations (17) of Art. 19 and develope the functions $\kappa v/r^3$, $\kappa\sigma/r^3$ in powers of δv, $\delta\sigma$ by putting $v = v_0 + \delta v$, $\sigma = \sigma_0 + \delta\sigma$, where δv, $\delta\sigma$ denote any of the series of inequalities to be found. This plan of procedure, which has some advantages, especially in the terms of lower orders in e, e', γ, $1/a'$, is outlined in the memoir referred to in Art. 274. The method, given above, for the determination of δg, together with its application to the other parts of the motion of the node and to the motion of the perigee, will be found in the same place.

Relations between the Higher Parts of the Motions of the Perigee and the Node and the Non-periodic Part of the Moon's Parallax.

288. The equation (41) is only one particular case of a much more general theorem by which all the parts of the motions of the perigee and of the node may be found when their principal parts—those depending on m only—have been obtained. Modified forms of the equation will serve for the determination of those parts of g which depend on *all* powers and products of e'^2, $1/a'^2$, e^2, γ^2, so that the motion of the node really depends only on the solution of an infinite determinant and of a series of simple linear equations with one unknown. The same is true of the perigee. For example, the part of the motion of the perigee which is of the order e^2 may be shown to be the value of δc given by

$$2\delta c \Sigma_j \left[(2j+1+m+c_0)\,\epsilon_j^{\,2} + (2j-1-m+c_0)\,\epsilon'^{2}_{-j}\right]$$

$$= \textit{Const. part, order } e^4, \textit{ in the expansion of } \kappa\,(X_{e^2}x_e + Y_{e^2}y_e)/a^2 R^3_{e^2} \ldots (42),$$

where x_0, x_e, x_{e^2} and y_0, y_e, y_{e^2} denote the parts in X and Y whose coefficients are of orders e^0, e^1, e^2, respectively, and where

$$X_{e^2}=x_0+x_e+x_{e^2}, \qquad Y_{e^2}=y_0+y_e+y_{e^2}, \qquad R^2_{e^2}=X^2_{e^2}+Y^2_{e^2}.$$

An important extension of the results (41), (42) consists in the fact that they are true when all powers of e'^2 are included in the various terms. In other words, when we suppose that all parts of the functions z_γ, R_{γ^2}, which depend on m, e'^2 and on γ^1, γ^2, have been found, and when g_0, K_j are replaced by their more accurate values $\mathrm{g}_0+e'^2f(\mathrm{m}, e'^2)$, $K_j+e'^2\gamma f'(\mathrm{m}, e'^2)$, the part of the value of g which depends on γ^2 and on all powers and products of m, e'^2 is given by (41); a similar result holds for part of the motion of the perigee given by (42). It is for this reason that the determination of the parts of the solution which depend only on m, e' should be the step immediately following the calculation of the intermediate orbit; the necessary parts of the motions of the perigee and of the node, which depend on m, e'^2, are found by the same method.

289. One general theorem is as follows :—Let e, γ be the constants of eccentricity and latitude, and let the coefficients ϵ_j, ϵ'_{-j}, K_j be expressed in terms of them; these coefficients are supposed to be of the form $ef(\mathrm{m}, e'^2)$, and c_0, g_0 of the form $\phi(\mathrm{m}, e'^2)$. The constant part of $1/r$ will contain the terms

$$(\mathrm{E}e^4+2\mathrm{F}e^2\gamma^2+\mathrm{G}\gamma^4)/\mathrm{a},$$

and the motions of the perigee and the node the terms

$$\mathrm{H}e^2+\mathrm{K}\gamma^2, \qquad \mathrm{M}e^2+\mathrm{N}\gamma^2,$$

respectively, where E, F, G, H, K, M, N are functions of m, e'^2 only. Put

$$e^2\mathrm{T}_e=2\frac{\mathrm{a}^3}{\kappa}\Sigma_j\left[(2j+1+\mathrm{m}+\mathrm{c}_0)\,\epsilon_j^2+(2j-1-\mathrm{m}+\mathrm{c}_0)\,\epsilon'^2_{-j}\right], \qquad \gamma^2\mathrm{T}_\gamma=4\frac{\mathrm{a}^3}{\kappa}\Sigma_j\,(2j+\mathrm{g}_0)^2K_j^2,$$

where powers of e'^2 are supposed to be included in c_0, ϵ_j, etc. *It may then be shown that*

$$\mathrm{HT}_e=6\mathrm{E}, \quad \mathrm{KT}_e=6\mathrm{F}=\mathrm{MT}_\gamma, \quad \mathrm{NT}_\gamma=6\mathrm{G} \qquad\qquad (43).$$

The theorem which lies at the basis of these results is as follows :—If X, Y, z (Art. 18) have been fully calculated to the order $e^p\gamma^{2q-p}$, where $p=0, 1, \ldots 2q$ (that is, to the order $2q$ in e, γ), the constant part of $1/r$ can be obtained to the order $e^p\gamma^{2q+2-p}$, where $p=0, 1, \ldots 2q+2$ (that is, to the order $2q+2$), without further reference to the equations of motion, by a purely algebraic formula involving only the values of X, Y, z to the order calculated. A different and more complete statement of this theorem is, that *the terms of order $2q$ with respect to e, γ, in the constant part of the expansion of $3/R_{2q}$ in powers of e, γ, are equal to the corresponding terms in the expansion of*

$$\left[\frac{3}{R_{2q-2}}-\frac{x_0X_{2q-2}+y_0Y_{2q-2}}{R^3_{2q-2}}-3\,\frac{z_1z_{2q-2}}{r_0^5}\,(x_0x_1+y_0y_1)\right],$$

where X_{2q}, Y_{2q}, Z_{2q} contain the terms as far as the order $2q$; x_{2q}, y_{2q}, z_{2q} are the terms in X_{2q}, Y_{2q}, z_{2q}, whose coefficients are of the order $2q$, and $R^2_{2q}=X^2_{2q}+Y^2_{2q}+Z^2_{2q-1}$. This result also holds when all powers of the solar eccentricity are included; the only quantity neglected is the solar parallax.

The proofs of the theorems in this and in the previous Article are too long to be inserted here; they will be found in Part II. of the author's paper referred to in Art. 274 above.

The first of Adams' two theorems* can be deduced from the result just enunciated, by putting $q=1$. It states that, in the constant part of the expression for a/r, there are no terms of the forms $e^2 f(\mathrm{m}, e'^2)$, $\gamma^2 f(\mathrm{m}, e'^2)$; here, $\mathrm{a}/(\mu/n^2)^{\frac{1}{3}}$ may be a function of m, e'^2, but it must not contain e, γ. His second theorem is immediately obtainable from the equations (43). It states that

$$\mathrm{H/K=E/F}, \qquad \mathrm{M/N=F/G}.$$

These interesting results may be also utilised as equations of verification.

* J. C. Adams, "Note on a remarkable property of the analytical expression for the constant term in the reciprocal of the Moon's radius vector," *M. N. R. A. S.* Vol. XXXVIII. pp. 460–472; *Coll. Works*, pp. 189–204.

CHAPTER XII.

THE PRINCIPAL METHODS.

290. A BRIEF account of the methods which have been used to attack the lunar problem, other than those considered above, will be given in this Chapter. In general, only those theories which have been tested by actual application to the discovery of the perturbations produced by the action of the Sun will be analysed. The historical order will be adhered to as far as possible, although the time elapsed between the first announcement of a plan of treatment and the full publication of the results, frequently makes it difficult to assign any exact date to a theory.

The history of the lunar theory and of celestial mechanics generally (in the sense in which these terms are now understood) began in 1687 with the publication of the *Principia*; the portion which especially refers to the motion of the Moon is contained in Props. 22, 25—35 of Book III. In this and in the later editions of the same work Newton succeeded in showing that all the principal periodic inequalities, as well as the mean motions of the perigee and the node, were due to the Sun's action, and he added some other inequalities which had not been previously deduced from the observations. The result which he obtains for the mean motion of the perigee is only half its observed value; it appears, however, from the Manuscripts* of Newton which have come down to us, that he had later succeeded in obtaining its value within eight per cent. of the whole. The conciseness of the proofs, when they are given, makes his work very difficult to follow. It is now generally recognized that he used his method of fluxions to arrive at many of the results, afterwards covering up all traces of it by casting them into a geometrical form; if this be so, the claim of Clairaut to be the first to apply analysis to the lunar theory must be somewhat modified. No substantial advance was made until the publication, more than sixty years later, of Clairaut's *Théorie de la Lune.*

* *Catalogue of the Portsmouth Collection of Books and Papers written by or belonging to Sir Isaac Newton.* Cambridge, 1888.

For the methods which Newton used, the reader is referred to the *Principia* and to the numerous commentaries which have been written on it. An analysis of the *Principia* and an account of Newton's life and works—published or in manuscript—together with many references, have been given by W. W. Rouse Ball*.

The history of the lunar theory up to the publication of the third volume of the *Mécanique Céleste* of Laplace and an account of the methods used, have been given by Gautier†. For further details concerning the theories of Newton, Clairaut, d'Alembert, the method given by Euler in the Appendix to his first theory, Euler's second theory, and concerning the theories of Laplace, Damoiseau and Plana, reference may be made to the *Mécanique Céleste* of Tisserand‡.

291. *Clairaut's Theory*§.

Clairaut commences by finding the differential equations of motion when the latitude of the Moon is neglected. He takes the inverse of the radius vector and the time as dependent variables, the true longitude as the independent variable, and he finally arrives at equations which are equivalent to (8), (9) of Art. 16; since the latitude is neglected, u_1 is, in his theory, the inverse of the radius vector.

The process of solution is one of continued approximation. He considers the orbit of the Moon to be primarily an ellipse, but, recognizing that the apse revolves quickly, he introduces the quantity denoted above by c, so that the first approximation is the modified ellipse. The forces due to the Sun's action are expressed in terms of the radii vectores of the Sun and the Moon and the difference of their longitudes, and thence, by means of the elliptic formulæ, in terms of the true longitude of the Moon—somewhat in the manner explained in Art. 127 above. The calculations are, in certain cases, carried to the second order of the disturbing forces, and, in particular, the motion of the perigee is obtained to this degree of accuracy.

The motion of the Moon in latitude is obtained by considering the equations for the variations of the node and the inclination; these correspond to the fifth and sixth of the equations of Art. 82. They are much less accurately worked out than the motion in the plane of the orbit.

Clairaut was the first to publish a method for the treatment of the lunar theory founded on the integration of differential equations. The artifice of modifying the first approximation by the introduction of c, in order to take the motion of the perigee into account from the outset, is also due to him. The determination of this motion is marked by an event

* *An Essay on Newton's Principia.* Macmillan, 1893.

† *Essai historique sur le Problème des Trois Corps.* Paris, 1817.

‡ Vol. III. Chaps. III.—VII., IX.

§ Announced in 1747. The first edition of the *Théorie de la Lune* was published in 1752, the second and most complete edition in 1765 at Paris. The latter is a quarto volume of 162 pages and it contains Tables for the calculation of the position of the Moon, founded on gravity only. The first set of tables was published separately in 1754.

which shows what progress astronomers had made in admitting the sufficiency of Newton's laws to account for all the known phenomena of the celestial motions. It must be remembered that Newton's later determination, as shown in his manuscripts, was then unknown.

It had long been a difficulty that the first approximation to the motion of the perigee gave only half the observed motion. Clairaut, thinking that the Newtonian law must be considerably in error, tried the effect of adding a term, of the form κ/r^3, to the forces resolved along the radius vector, and he succeeded in showing that it would account for the observed motion. It then occurred to him to carry his approximation a step further, with the law of the inverse square only, and to see what effect the evection had when introduced into the expressions for the disturbing forces. He soon found that the new term was nearly equal in absolute value to the second term and that the greater part of the motion was thus accounted for by the law of Newton (see Art. 169).

292. *D'Alembert's Theory* *.

This is very similar in its general plan to that of Clairaut, but while the latter worked out his results numerically, d'Alembert considered a literal development and carried out his computations with more completeness. He gave the term $\frac{225}{32}m^3$ in the motion of the perigee: Clairaut had only obtained its numerical value. His method of approaching the modified orbit is much more logical; he introduces a part of the Sun's action into the first approximation by proceeding in a manner analogous to that of Art. 67 above.

D'Alembert made several contributions to the theory. He succeeded in showing that terms increasing continually with the time can be avoided, and he gave a direct method of approaching the first approximation. He also recognized the fact that the question of the convergence of the series obtained ought not to be neglected. He further considered the effects produced by small divisors and showed that the coordinates might be expressed by means of only four arguments which were necessarily related to the orders of the coefficients. Tables are added at the end of his researches.

293. *Euler's First Theory* †.

Euler commences by considering the equations of motion referred to cylindrical coordinates; these, translated into modern notations, are the equations given at the beginning of Art. 16. The equation for the latitude is immediately replaced by two others which are practically those for the variations of the node and inclination, and they are obtained under the assumption that the first differential of $r \tan U$, with respect to the time, has the same form in disturbed and undisturbed motion. The second equation of motion is integrated and the resulting value of $\dot{v}$ is substituted in the first equation which then contains only the differentials of r_1, with respect to the time, and certain integrals depending on the forces.

* *Recherches sur différents Points importans du Système du Monde.* Pt. I. (1754) *Théorie de la Lune*, 8vo. LXVIII. + 260 pp. D'Alembert sent in his theory to the Secretary of the Academy in January, 1751.

† *Theoria Motus Lunae*, etc. (with an Appendix) 4to. 347 pp. Petrop. 1753.

The independent variable is changed from t to the true anomaly f, and for r is put its elliptic value (in terms of f) multiplied by $1+\nu$; ν is then a small quantity depending on the disturbing forces (cf. Hansen's method, Art. 209). Euler thus arrives at a differential equation of the second order in which ν is the dependent and f the independent variable.

After expanding the disturbing forces according to powers of e, etc., he divides them into classes:—those independent of e, e', i; those independent of e', i; and so on, after the manner explained in Chaps. VII., XI. In considering the inequalities of the second class (Art. 132), he finds

$$v = \text{const.} + c_1 f + \text{periodic terms};$$

so that the motion of the perigee depends on c_1. This is not determined directly. Its observed value is used and Euler then compares the latter with the value deduced from theory in order to test the Newtonian law. He has previously assumed that the attraction between the Earth and the Moon is of the form $\mu/r^2 - \text{const.}$; the constant is shown to be very small and well within the limits of error caused by the neglect of higher terms in the approximations.

The numerical values of the constants to be used in the theory—which is numerical as far as m is concerned and algebraical in respect of the other constants—are determined by the consideration of thirteen eclipses; the want of definiteness in the meanings to be assigned to the constants, which affected the results of Clairaut and d'Alembert, is avoided, for Euler uses the formulæ of his own theory in the calculation of these eclipses.

In the Appendix, an investigation, which practically amounts to the method of the variation of arbitrary constants, is given and worked out with some detail. Euler expresses himself as unsatisfied with both the theories he has explained.

294. *Euler's Second Theory**.

The method which Euler has here set forth with much detail is interesting as the first attempt to employ rectangular coordinates referred to moving axes in the Lunar Problem. He considers an axis of x revolving with the mean angular velocity of the *Moon* in the Ecliptic, that of y being also in this plane and that of z perpendicular to the plane. Taking the mean distance of the Moon as a, its coordinates are assumed to be $a(1+x)$, ay, az, so that x, y, z are small quantities depending on the solar action and on the lunar eccentricity and inclination; a is defined to be such that x contains no constant term. The equations of motion are found in the usual way, the disturbing

* *Theoria Motuum Lunae, nova methodo pertractata una cum Tabulis astronomicis unde at quodvis tempus Loca Lunae expedite computari possunt,...*: J. A. Euler, W. L. Krafft, J. A. Lexell. Opus dirigente L. Eulero. 4to. 775 pp. Petrop. 1772.

forces being developed in powers of $1/r'$. It is to be noticed that this method, like that of Chap. XI., has the advantage of allowing the disturbing forces to be expressed as homogeneous functions of x, y, z of the first, second, and higher degrees; but the relation between the degrees of the homogeneous functions and those of $1/a'$, which was observed in the equations of Section (iii), Chap. II., does not hold in Euler's method: the coefficients of these functions, in Euler's theory, are expressible by means of the two arguments ξ, w'.

The independent variable is w', and Euler puts

$$n/n' = m_1 + 1;$$

so that $1 + m_1$ is the ratio of the mean motions of the Moon and the Sun: observation gives $m_1 = 12{\cdot}36\ldots$. The forces $a(1+x)/r^3$, ay/r^3, az/r^3, due only to the mutual actions of the Earth and the Moon, are expanded in powers of x, y, z.

The general solution is then supposed to be of the form

$$x = A + eB_1 + e^2B_2 + \ldots + e'C_1 + e'^2C_2 + \ldots + ee'C_{11} + \ldots + i^2D_2 + \ldots,$$

with similar expressions for y, z. Here e, i are the two arbitrary constants of the solution corresponding to the eccentricity and the inclination. Substituting these values in the differential equations and equating the coefficients of the various powers and products of e, e', etc. to zero, he obtains a series of differential equations for the determination of A, B_1, ... The parts dependent on A give the variational inequalities, those dependent on B_1, B_2,... the elliptic inequalities, and so on. In the determination of B_1, the motion of the perigee arises; as in his earlier methods, he assumes its value from observation and verifies his results by means of the calculated value. The motion of the node is treated in the same manner. The various differential equations are solved by the method of indeterminate coefficients.

M. Tisserand remarks that Euler's method of dividing up the inequalities into classes requires some modification when we proceed to terms of higher orders, owing to the fact that the motions of the perigee and the node contain powers of e^2, e'^2,... ; the arguments depending on these motions, when expanded so as to put the solution into Euler's form, would introduce into the coefficients terms depending on the time. A reference to Art. 283 above, will show how this objection to Euler's method may be removed.

Euler's main contributions to the lunar theory are:—the application of moving rectangular axes; the method of the variation of arbitrary constants, as given in the appendix to his first theory; the use of indeterminate coefficients in the solution of the differential equations; a new method for the determination of the constants from observation; the formation and solution of equations of condition to determine the constants from observation when the number of unknowns is less than the number of equations; the final expression of the coordinates by means of angles of the form $\alpha + \beta t$. He also added to the subject in many other directions, and much of the progress which has since been made, may be said to be founded on his results.

295. *Laplace's Lunar Theory* *.

The publication of Laplace's *Mécanique Céleste* marked a new epoch in the history of the lunar theory, owing to the general plan of treatment adopted and to the manner in which it was carried out. Some account of Laplace's method has already been given in previous chapters. In Section (ii) of Chap. II. his general equations of motion—with the *true* longitude as independent variable and with the time, the inverse of the projected radius vector and the tangent of the latitude as dependent variables—have been obtained. The first approximation—found by neglecting the action of the Sun—has been given in Art. 52, and the manner in which this is modified to prevent the occurrence of terms proportional to the time, in Art. 70. By means of the modified ellipse, those parts of the equations of motion which are due to the action of the Sun are expressed in terms of the true longitude (Art. 127).

The equations can then be integrated. Laplace's method is to assume the solution to be a sum of periodic terms whose coefficients are unknown, and to substitute it in the differential equations: in each unknown coefficient the characteristic is written separately; he thus obtains, on equating the coefficients of the different periodic terms to zero, a series of equations of condition by means of which the coefficients can be calculated. The new values of u_1, t, s are then used to find the third approximation. The method of procedure is similar to that of Chap. VII., except that, instead of the equation for z, the second of the equations (11), Art. 16, is used and solved in the same manner as the first of these equations; the analysis is, however, rather more simple owing to the forms of the left-hand members of the equations for u_1, s. Terms up to the second order in e, e', γ, a/a' are considered and certain terms of higher orders whose coefficients become large, owing to small divisors, are also included. The approximations are, in general, taken to the second order of the disturbing forces.

The coefficients are not developed in powers of m. As soon as the equations giving the values of c, g and the equations of condition between the coefficients have been obtained, Laplace substitutes the numerical values of the constants in all terms; only the characteristics are left arbitrary, so that a small change in the numerical values of any of the constants, except m, will not sensibly affect the coefficients. The theory is therefore a semi-algebraical one. The value of the coefficient of the principal elliptic term in the expression of the *mean* longitude in terms of the *true* being thus obtained, the value of e necessary to his theory is deduced from observation; the constant γ is found in a similar manner. His constants e, γ are such that the

* *Mécanique Céleste*, Pt. II. Book VII. pp. 169—303, 4to. Paris, 1802. Several editions have since appeared; the latest, now in the course of publication, is in a collection of all Laplace's works. Laplace's investigations cover a period of thirty years anterior to the publication of Vol. III. of the *Méc. Cél.*

coefficient of the principal elliptic term in the expression of u_1 in terms of v and that of the principal term of s in terms of v, are the same as in undisturbed motion *.

Finally, the numerical values of the constants are all substituted, and a reversion of series gives u_1, v, s and thence $1/r$, v, U in terms of the time †.

296. Besides giving a general treatment of the Lunar Theory, Laplace enriched the subject with several new discoveries. Of these, the most noted is his explanation of the cause of the secular acceleration of the Moon's mean motion‡—a phenomenon which had been observed many years before and which had been the subject of several prizes offered by various academies. Laplace, after an attempt to account for it by supposing that a finite time was necessary for the transmission of the force of gravity, announced in 1787 that it was due to a slow variation in the eccentricity of the Earth's orbit, and his theoretical determination agreed almost exactly with the value deduced from the observations. He also showed that the same cause produced sensible accelerations in the motions of the node and the perigee; his results were confirmed by a later examination of the observations.

A curious fact concerning the discovery of the cause of the secular acceleration of the mean motion, the theoretical value of which remained unquestioned for over sixty years, was pointed out by J. C. Adams §. Laplace and his followers had integrated the equations of motion as if e' were constant, substituting its variable value in the *results*, and had then determined the acceleration to a high degree of approximation. Adams showed that, although this method of procedure is permissible in a first approximation, it is necessary to introduce the variability of e' into the differential equations themselves when proceeding to higher orders. He then found that the true theoretical value, which amounted to about 6″ per century in a century, was only a little more than half of the value obtained by Laplace and Plana and therefore that theory was insufficient to account completely for the observed value. A controversy, which lasted for several years, arose concerning the validity of Adams' method; his value was, however, confirmed at various times by several investigators amongst whom may be mentioned Delaunay ||, Plana ¶, Lubbock **, Cayley †† and Hansen ‡‡. It must be stated, however, that doubts have been raised concerning the correctness of the value deduced from observation by the researches of Prof. Newcomb §§ into ancient eclipses. The question turns chiefly on the trustworthiness of the records. A full discussion of the points at issue is given by Tisserand || ||.

* See Art. 159 above.

† A portion of Laplace's second approximation and the determinations of c, g to the order m^3 are given by H. Godfray, *Elementary Treatise on the Lunar Theory*.

‡ See Arts. 319–322 below.

§ "On the Secular Variation of the Moon's Mean Motion," *Phil. Trans.* 1853, pp. 397—406; *M. N. R. A. S.* 1853; *Coll. Works*, pp. 140—157.

|| "Sur l'accélération séculaire du moyen mouvement de la Lune," *Comptes Rendus*, Vol. XLVIII. pp. 137—138, 817—827.

¶ "Mémoire sur l'équation séculaire de la Lune," *Mem. d. Accad. d. Sc. di Torino*, Vol. XVIII. pp. 1—57.

** "On the Lunar Theory," *Mem. R. A. S.* Vol. XXX. pp. 43—52.

†† "On the Secular Acceleration of the Moon's Mean Motion," *M. N. R. A. S.* Vol. XXII. pp. 171—231; *Coll. Works*, Vol. III. pp. 522—561.

‡‡ "Sur la controverse relative à l'équation séculaire de la Lune," par M. Delaunay, *Comptes Rendus*, Vol. LXII. pp. 704—707.

§§ "Researches on the Motion of the Moon," *Washington Observations*, 1875, pp. 1—280.

|| || *Mécanique Céleste*, Vol. III. Chaps. XIII, XIX.

297. *The Theory of Damoiseau**.

Damoiseau follows Laplace's method almost exactly. He assumes the final forms of the expressions for u_1, nt, s in terms of v and substitutes them directly in the differential equations. A number of equations of condition, involving the unknown coefficients in a more or less complicated manner, are thus obtained and these are solved by continued approximation. Numerical values are used all through and the theory is therefore entirely numerical. When the coefficients have been obtained, a reversion of series is made in order to express the coordinates in terms of the time; this is also done by the use of indeterminate coefficients—a method always available when the arguments of the required series are known.

The object of the theory appears to be the determination of the coefficients accurately to one-tenth of a second of arc. For this purpose he carries them to the hundredth of a second and includes certain sensible terms due to the actions of the planets and to the figure of the Earth. The results are given very concisely, but the work will be easily followed after a perusal of Laplace's theory as given in the *Mécanique Céleste*. The labour of finding the values of the coefficients may be grasped from the fact that the mere writing down of the equations of condition occupies half the Memoir. The tables which he deduced† from the results of this theory were not entirely disused until those of Hansen appeared.

298. *The Theory of Plana‡*.

This is an extension of a theory worked out by Plana and Carlini and sent in to compete for a prize offered by the Paris Academy of Sciences in 1820. A prize was awarded to them and also to Damoiseau for his theory. The results of Plana and Carlini were not printed, but later Plana issued the three large volumes referred to in the footnote. The method of Laplace is used; Plana, however, instead of substituting numerical values, makes a literal development in powers of m, e, e', γ, a/a'. The results are, in general, carried to the fifth order of small quantities; certain coefficients, which are expressed by slowly converging series, are carried to the sixth, seventh and eighth orders. In point of accuracy, judged by Hansen's theory, it is about equal to that of de Pontécoulant and slightly inferior to the numerical theory of Damoiseau; the inferiority is partly due to the slow convergence of the series arranged in powers of m and partly to errors which have crept into the work—errors unavoidable where the developments are of such length and complexity. As a literal development it has only been completely superseded by Delaunay's theory.

* "Mémoire sur la Théorie de la Lune," *Mém.* (*par divers savants*) *de l'Inst. de France*, Vol. I. (1827), pp. 313—598.

† *Tables de la Lune, formées par la seule théorie de l'attraction et suivant la division de la circonférence en* 400 *degrés*, Paris, 1824. *Tables...en* 360 *degrés*, Paris, 1828.

‡ *Théorie du Mouvement de la Lune*, 8vo. Turin, 1832, Vol. I. 794 pp.; II. 865 pp.; III. 856 pp.

299. *The Method of Poisson**.

Poisson proposed to apply the method of the Variation of Arbitrary Constants to the solution of the lunar problem. For this purpose he introduces the equations of Art. 83 above. The disturbing function is to be expanded by the purely elliptic values of the coordinates and the result substituted in the right-hand members of the equations. To obtain the second approximation to the values of the elements, they are regarded at first as constants in the right-hand members; the equations may then be solved and the resulting values of the elements, in terms of the time, are to be used as the basis of a second approximation by substituting them, instead of their constant values, in the same parts of the equations. To obtain the solar inequalities in the Moon's motion, the method in this form is almost useless on account of the enormous developments which it would entail, and it would not be considered here were it not for its value in investigating the inequalities arising from other sources and, in particular, for those inequalities known as 'long-period' and 'secular.' In fact, Poisson only gives a few calculations as illustrations of the method. It is chiefly of value in the planetary theory.

300. *The Method of Lubbock†*.

The publication of Lubbock's researches in the volumes of the *Phil. Trans.* between 1830 and 1834, places his method next in historical order; they are collected and extended in the pamphlets referred to in the footnote. The method is the same as that of de Pontécoulant, whose results were not published until 1846; but, from the remarks made by Lubbock and de Pontécoulant in their prefaces, it is evident that they had adopted the same plan independently. Lubbock never carried out his method with any completeness: his published papers contain an explanation of the method, a full development of the second approximation, and the calculation of the earlier approximations to the coefficients of certain classes of terms; his results are compared with those of Plana.

Next in order come the theories of Hansen and Delaunay which have been already treated. Finally, mention must be made of Airy's method‡, which was rather a verification of previous results than a complete theory in itself.

Airy proposed to take Delaunay's expressions after numerical values had been substituted for the constants and, considering each coefficient to need a small unknown correction, to substitute the results, together with the unknown parts, in the equations of

* "Mémoire sur le mouvement de la Lune autour de la Terre," *Mém. de l'Acad. des Sc. de l'Inst. de France*, Vol. XIII. (1835) pp. 209—335. (Read in 1833.)

† *On the Theory of the Moon and on the Perturbations of the Planets*, London, 8vo. Pt. I. (1834), 115 pp., with an Appendix containing Plana's results; Pts. II. (1836), III. (1837), IV. (1840), 417 pp.; Pt. X. (1861), 94 pp., with tables.

‡ *Numerical Lunar Theory*, London, 1886, fol. 178 pp.

motion*. He had worked at this for several years but, after the volume containing his results was published, he discovered a serious omission which altogether invalidated them; the large corrections which he had found were necessary to make Delaunay's results satisfy the equations of motion, were probably due to this unfortunate error. In a letter† to the Secretary of the Royal Astronomical Society, he says, "I keep up my attention to the general subject, but with my advanced age (eighty-eight) and failing strength I can scarcely hope to bring it to a satisfactory conclusion. I will only further remark that I believe the plan of action which I had taken up would, if properly used, have led to a comparatively easy process, and might in that respect be considered as not destitute of all value."

301. *Tables.*

The tables of the Moon's motion which have been formed from the results of theory alone, in order to calculate the position of the Moon at any time, have already been referred to, in connection with the theories from which they were deduced. In addition, we may mention those of Mayer (London, 1770) formed by a combination of theory and observation, of Mason (London, 1787), which were Mayer's tables improved, of Burg (Paris, 1806), of Burckhardt (Paris, 1812) and, for the Parallax of the Moon, of Adams (*M. N. R. A. S.* Vol. XIII. 1853; *Nautical Almanac*, 1856; *Coll. Works*, pp. 89—107). Later efforts in this direction have been made chiefly for the purpose of correcting Hansen's tables (see Art. 238).

302. In making a comparison of the various methods of treating the lunar problem, several considerations enter. There does not appear to be any method which is capable of furnishing the values of the coordinates with a degree of accuracy comparable with that of observation, without great labour; and, in the present state of the lunar theory, looking only to a practical issue, what is required is rather a verification of the results of previous methods, say those of Hansen and Delaunay, than new developments. Again, some methods appear to be most effective for one class of inequalities while other methods give another class of inequalities most accurately. The question to be discussed is mainly the relation between the accuracy obtained and the labour expended.

As regards the inequalities produced by the action of the Sun, the methods may be divided into three classes. The first or algebraical class contains those in which all the constants are left arbitrary; the second or numerical, those in which the numerical values of the constants are substituted at the outset; the third or semi-algebraical, those in which the numerical values of some of the constants are substituted at the outset, the others being left arbitrary: the most useful case of the last class appears to be that in which the numerical value of the ratio of the mean motions is alone substituted. The advantage of an algebraical development will be readily recognized. In a numerical development, slow convergence is to a great extent avoided, but the source of an error is traced with great difficulty and any change in the values of the arbitraries can not be fully accounted for without an extended recalculation. The semi-algebraical class, in which the value of m is alone substituted, appears to possess an accuracy nearly equal to that of a numerical development, and it has the advantage of leaving those constants arbitrary whose values are known with least accuracy.

It is difficult to judge of the labour which any particular method will entail, without performing a considerable part of the calculations by that and by other methods. As far

* An exhaustive analysis and criticism of Airy's method is given by M. Radau, *Bull. Astronomique*, Vol. IV. pp. 274—286.

† "The Numerical Lunar Theory," *M. N. R. A. S.* Vol. XLIX. p. 2.

as it is possible to estimate, either by general considerations or by the amount of time previous lunar theorists have spent over their calculations, it may be stated that those methods which have the true longitude as the independent variable must be altogether excluded if the solar perturbations are required, owing to the necessary reversion of series. For a *complete* algebraical development carried to a greater accuracy than that of Delaunay, none of the methods given up to the present time seem available without the expenditure of enormous labour: Delaunay's calculations occupied him for twenty years. If we may judge from the inequalities computed up to the present time, the methods of Chap. XI. seem to be best suited to a numerical or semi-algebraic development. It is true that they give the results expressed in rectangular instead of in polar coordinates, but the labour of transformation is not excessive in comparison with that expended on the previous computations, while the accuracy obtained far surpasses that of any other method; the transformation of the series, however, would not be necessary for the formation of tables. The disadvantage of de Pontécoulant's method is the necessity of obtaining the parallax, with an accuracy much beyond that required for observation, before the longitude can be found; this remark applies also to the methods of Chap. XI., but in rather a different way. Hansen's method labours under the disadvantage of putting the results under a form which makes comparison with those of other methods difficult. Another consideration which is a powerful factor, is the question as to how far the ordinary computer, who works by definite rules only, can be employed in the calculations; and here the methods of Chap. XI. appear to have an advantage not possessed by any of the earlier theories.

With reference to the classical treatises on Celestial Mechanics, there is little doubt that the works of Euler and Laplace will best repay a careful study; those of Lagrange in a different direction—the general problem of three bodies—must also be mentioned. The ideas upon which all the later investigations have been built, may be said to have originated from the works of one or other of these three writers.

CHAPTER XIII.

PLANETARY AND OTHER DISTURBING INFLUENCES.

303. An explanation of the way in which the principal effects of planetary action and of the figure of the Earth may be included in the lunar theory will be given in this chapter. A general plan of integration for the new terms introduced into the disturbing function will be first explained; the discovery and development of the disturbing functions for the direct and indirect actions of the planets and for the direct effect of the ellipticity of the Earth then follow, the results being illustrated by applying them to a few of the principal inequalities. The perturbations produced by the motion of the ecliptic and by the secular variation of the solar eccentricity are, owing to their peculiar nature, treated by special methods. In all cases, the developments will be only given as far as they are necessary for the purpose of explanation; references are given to the memoirs in which more complete investigations may be found. As far as the end of Art. 318, Delaunay's notation will be used; the determination of the secular acceleration being made by the use of de Pontécoulant's equations, we use the notation of Chap. VII. in Arts. 319—322.

The effect of the terrestrial Tides and of the figure of the Moon on the motion of the latter will not be treated here. The former is considered in the Memoirs of Prof. G. H. Darwin* in detail; the chief effect is on the Moon's mean period and mean distance, and the amount of the correction, within the limits of time during which observations have been recorded, is very small. As to the latter, it is very doubtful whether it produces any appreciable effect: Hansen introduces an empirical periodic term supposed to be due to the difference between the centre of mass and the centre of figure of the Moon †.

General ethod of Integration.

304. The expression of the disturbing causes which affect the motion of the Moon can, in nearly all cases, be made by inserting additional periodic and constant terms in the disturbing function. The periods and coefficients of these terms of the disturbing function—the parts which involve the elements

* *Phil. Trans.* 1879—1881.

† *Darlegung*, I. pp. 175, 474—479.

of the Moon as well as those arising from other sources—can generally be found with an accuracy sufficient for practical purposes; for this reason, it is advisable to use a method of integration which shall be adaptable easily to any periodic term, and such a method, founded on Delaunay's formulæ, has been devised by Dr Hill*. Its value chiefly depends on the fact that the coefficients of the new terms in the disturbing function are always small and that, in consequence, it is seldom necessary to consider the changes produced in the new terms of the disturbing function by those changes of the elements which occur when any one of the old or new periodic terms is eliminated by Delaunay's processes. The operations are therefore similar to the majority of those mentioned in the last paragraph of Art. 197, but it will be seen that we may use numerical values for the elements of the Moon's orbit and that, owing to this fact, the operations of Delaunay may be very considerably abridged. The numerical results given below are those obtained by Radau† in a valuable Memoir to which frequent reference will be made. He introduces a slight modification of Hill's method and his numerical values for δM_0 differ to a small extent from those of Hill. Periodic terms only will be discussed; the changes produced by new non-periodic terms due to the inequalities considered below, are very small.

It is supposed that the periodic terms, arising solely from the action of the Sun considered to be moving in an elliptic orbit, have been eliminated, and that the disturbing function contains only the remaining constant portion together with the new terms to be considered. It is further supposed that the operation of Art. 198 above has not been carried out, and that the final change of constants, which Delaunay makes in order to reduce his expressions to a suitable form (Art. 200), is as yet not performed. The results required here are all contained in Delaunay's volumes: the latter will be referred to as in Chap. IX.

305. Delaunay's canonical equations are (Art. 183)

$$\frac{dL}{dt}=\frac{\partial R}{\partial l},\ \ldots,\ \ldots;\quad \frac{dl}{dt}=-\frac{\partial R}{\partial L},\ \ldots,\ \ldots.$$

Let $-B$ be the constant part of R which remains after the periodic terms due to the Sun have been eliminated; we then have, by Art. 198,

$$l_0=\frac{\partial B}{\partial L},\quad g_0=\frac{\partial B}{\partial G},\quad h_0=\frac{\partial B}{\partial H}.$$

* It is contained in pt. III. of his Memoir "On certain Lunar Inequalities due to the action of Jupiter and discovered by Mr E. Neison," *Astron. Papers for Amer. Eph.* Vol. III. pp. 373—393.

† "Recherches concernant les Inégalités planétaires du Mouvement de la Lune," *Ann. de l'Obs. de Paris (Mémoires)*, Vol. XXI. pp. 1—114. See also, "Remarques sur certaines inégalités à longue période du mouvement de la Lune," *Bulletin Astronomique*, Vol. IX. pp. 137—146, 185—212, 245—246.

Let δR be the new part of R, and let δL, δG, δH, δl, δg, δh be the new parts of L, G, H, l, g, h, due to δR; let δl_0, δg_0, δh_0 be the new parts of l_0, g_0, h_0 (which are functions of L, G, H), due to δL, δG, δH. The canonical equations may be written,

$$\frac{d}{dt}\delta L = \frac{\partial}{\partial l}\delta R, \ldots, \ldots; \quad \frac{d}{dt}\delta l = \delta l_0 - \frac{\partial}{\partial L}\delta R, \ldots, \ldots \quad \ldots\ldots\ldots\ldots(1).$$

We choose out one of the periodic terms of δR and put

$$\delta R = + A \cos(il + i'g + i''h + \alpha t + \beta) = A\cos\theta,$$

where $\alpha t + \beta$ is the part of the argument independent of the lunar elements, and where A is the coefficient, containing three of the six lunar elements, namely, L, G, H or a, e, γ. We have

$$\theta = (il_0 + i'g_0 + i''h_0 + \alpha)\,t + \beta' = Mt + \beta',$$

suppose, so that M is the motion of the argument θ.

As A is always very small, we can, with a sufficient approximation, substitute the value of δR in (1) and integrate on the supposition that L, G, H, l_0, g_0, h_0 are constant when multiplied by the small quantity A. In this way we find

$$\delta L = \frac{i}{M}A\cos\theta, \quad \delta G = \frac{i'}{M}A\cos\theta, \quad \delta H = \frac{i''}{M}A\cos\theta.$$

Whence, since a, e, γ, l_0, g_0, h_0 are functions of L, G, H,

$$\delta a = \left(i\frac{\partial a}{\partial L} + i'\frac{\partial a}{\partial G} + i''\frac{\partial a}{\partial H}\right)\frac{A}{M}\cos\theta, \quad \delta e = \ldots, \quad \delta\gamma = \ldots \quad \ldots\ldots(2);$$

$$\delta l_0 = \left(i\frac{\partial l_0}{\partial L} + i'\frac{\partial l_0}{\partial G} + i''\frac{\partial l_0}{\partial H}\right)\frac{A}{M}\cos\theta, \quad \delta g_0 = \ldots, \quad \delta h_0 = \ldots.$$

The second three of equations (1) give

$$\frac{d}{dt}\delta l = \delta l_0 - \frac{\partial A}{\partial L}\cos\theta, \quad \ldots, \quad \ldots.$$

Substituting the values of δl_0, δg_0, δh_0 and integrating on the supposition that the lunar elements are constant in the right-hand members, we find

$$\delta l = \left(i\frac{\partial l_0}{\partial L} + i'\frac{\partial l_0}{\partial G} + i''\frac{\partial l_0}{\partial H}\right)\frac{A}{M^2}\sin\theta - \frac{\partial A}{\partial L}\frac{\sin\theta}{M}, \quad \delta g = \ldots, \quad \delta h = \ldots \quad \ldots(3).$$

The equations (2), (3) give the new terms to be added to the elements.

306. As regards the calculation of the various quantities present in (2), (3), the partial differential coefficients of a, e, γ, l_0, g_0, h_0 with respect to L, G, H may be obtained from the expressions given by Delaunay*. Since

* Delaunay, II. pp. 235—238.

we shall not consider the changes produced in R by the changes in the elements, the numerical values of n'/n, e, e', γ, a/a' may be substituted in the results: the numerical values of the lunar constants will not be quite the same as those used by Delaunay in his final results, because the final transformation which alters the meaning of n, e, γ, a (Art. 200) has not been made; the necessary modifications can be obtained from the formulæ given by Delaunay for the transformations*. The values thus obtained by Radau are

$$n'/n = 0{\cdot}0744, \quad e = 0{\cdot}0549, \quad \gamma = 0{\cdot}0449, \quad a/a' = 0{\cdot}00257.$$

To obtain the partial derivatives of A with respect to L, G, H, we have

$$\frac{\partial A}{\partial L} = \frac{\partial A}{\partial a}\frac{\partial a}{\partial L} + \frac{\partial A}{\partial e}\frac{\partial e}{\partial L} + \frac{\partial A}{\partial \gamma}\frac{\partial \gamma}{\partial L}, \quad \ldots, \quad \ldots,$$

in which the partials $\partial a/\partial L, \ldots$ may be numerically calculated in the manner just explained. These calculations being made once for all, we can obtain very simple formulæ for the determination of the coefficient of any periodic term.

Instead of g, the mean longitude $M_0 = l + g + h$ is introduced, so that

$$\delta M_0 = \delta l + \delta g + \delta h;$$

and instead of M, the ratio $p = n'/\mathrm{M}$. Put

$$A' = \frac{pA}{n'^2 a^2}, \quad a\frac{\partial A}{\partial a} = jA, \quad e\frac{\partial A}{\partial e} = j'A, \quad \gamma\frac{\partial A}{\partial \gamma} = j''A.$$

After inserting the numerical values of all the known terms, as explained above, Radau finds† that the equations (2), (3) become

$$\left.\begin{aligned}
\delta a/a &= (0{\cdot}14901 i - 0{\cdot}000246 i' - 0{\cdot}000006 i'')\, A' \cos\theta,\\
\delta e &= (1{\cdot}4215\, i - 1{\cdot}4238\, i' + 0{\cdot}00024\, i'')\, A' \cos\theta,\\
\delta \gamma &= (0{\cdot}00010 i + 0{\cdot}41203\, i' - 0{\cdot}41370\, i'')\, A' \cos\theta,\\
\delta M_0 &= \{(-3{\cdot}0576\, i + 0{\cdot}05601\, i' - 0{\cdot}01124\, i'')\, p\\
&\quad - 0{\cdot}14876 j + 0{\cdot}02551\, j' + 0{\cdot}03492\, j''\}\, A' \sin\theta,\\
\delta l &= \{(-3{\cdot}0826\, i + 0{\cdot}06142\, i' - 0{\cdot}03621\, i'')\, p\\
&\quad - 0{\cdot}14901 j - 25{\cdot}891\, j' - 0{\cdot}00232\, j''\}\, A' \sin\theta,\\
\delta h &= \{(-0{\cdot}03641 i + 0{\cdot}02890\, i' - 0{\cdot}00372\, i'')\, p\\
&\quad + 0{\cdot}000006 j - 0{\cdot}00435\, j' + 9{\cdot}2169\, j''\}\, A' \sin\theta
\end{aligned}\right\} \ldots\ldots\ldots (4).$$

The simplicity of these equations enables us to calculate easily the first approximation (generally sufficient), according to powers of A or A', to the coefficient of any term. Examples will be found below.

* Delaunay, II. p. 800.

† *Recherches* etc., p. 36.

307. To calculate the corresponding terms in the coordinates, it will often be sufficient to limit their expressions to the principal elliptic and solar terms, and to find the small changes in the coordinates induced by the calculated changes in the elements, after inserting the numerical values of all parts of the coefficients except the characteristics. For example, it is in many cases sufficient to put

$$v = M_0 + 2e \sin l + 0{\cdot}41 e \sin (2D - l),$$

where the numerical part 0·41 arises from the series in powers of n'/n etc. From this expression, δv may be obtained by putting $D = M_0 - n't - \epsilon'$, and causing M_0, e, l to receive the small increments calculated above.

If this be not sufficiently approximate, we can take the principal terms in the unreduced values of the coordinates in terms of the elements* and submit them to a variation δ, all the elements being supposed variable.

General method for the Inequalities produced by the Direct and Indirect Actions of the Planets.

308. *The Disturbing Functions.*

Let x', y', z', r' be the coordinates of the Sun, x, y, z, r those of the Moon, referred to axes fixed in direction and passing through the Earth, and let S be the cosine of the angle between r, r'. The disturbing function for the motion of the Moon, due to the Sun, is, by Art. 107,

$$R = \frac{m'r^2}{r'^3}\left(\tfrac{3}{2}S^2 - \tfrac{1}{2}\right) + \frac{m'r^3}{r'^4}\left(\tfrac{5}{2}S^3 - \tfrac{3}{2}S\right) + \dots.$$

In order that the coordinates x', y', z', r' may be considered to refer to the motion of the Sun about G (the centre of mass of the Earth and the Moon), it is necessary to multiply the second term of this expression by

$$(E - M)/(E + M).$$

Let ξ, η, ζ, D be the coordinates and distance of a planet P, referred to the same axes. The action of a planet of mass m'', on the motion of the Moon, will evidently be expressed by a disturbing function of the same form as R, namely R', where

$$R' = \frac{m''r^2}{\mathrm{D}^3}\left(\tfrac{3}{2}S'^2 - \tfrac{1}{2}\right) + \frac{m''r^3}{\mathrm{D}^4}\left(\tfrac{5}{2}S'^3 - \tfrac{3}{2}S'\right) + \dots:$$

S' being the cosine of the angle between r, D. In order that ξ, η, ζ, D may be considered to refer to G as origin, it is necessary to multiply the second term by $(E - M)/(E + M)$.

* Delaunay, II. Chaps. VII.—IX.

The ratios r/r', r/D, m''/m' are always small and, in most cases, the effect of R' on the motion of the Moon will be sufficiently accounted for by considering only the first term of R'; the other terms will therefore be neglected here. The inequalities produced by R' are said to be due to the *direct* action of the planets. To each planet will correspond a function R'; but since the terms produced by the combination of two terms, one from each such function, are generally negligible, it is only necessary to consider one of these functions, applying it to the case of each planet successively.

The solar inequalities, as far as they arise from the purely elliptic motion of the Sun, are supposed to have been determined. The actions of the planets on the motion of the Earth produce small deviations from elliptic motion in the apparent motion of the Sun: these, being substituted in R, may be considered as small corrections $\delta x'$, $\delta y'$, $\delta z'$ to the coordinates x', y', z'. As these corrections are never large, it will be sufficient, for the inequalities thus produced in the motion of the Moon, to limit R to its first term. The lunar inequalities arising in this way are said to be due to the *indirect* action of the planets. Since m'' is very small compared with m', it will not be necessary to consider these variations of x', y', z' in R'. See Art. 310 (*d*).

309. *Separation of the terms in R, R', and their expressions in polar coordinates.*

Confining R, R' to their first terms we have, on introducing rectangular coordinates,

$$\frac{1}{m'}R = \tfrac{3}{2}\frac{(xx'+yy'+zz')^2}{r'^5} - \tfrac{1}{2}\frac{r^2}{r'^3},$$

$$\frac{1}{m''}R' = \tfrac{3}{2}\frac{(x\xi+y\eta+z\zeta)^2}{D^5} - \tfrac{1}{2}\frac{r^2}{D^3}.$$

These may be written

$$\frac{1}{m'}R = \frac{r^2-3z^2}{4}\left(\frac{1}{r'^3}-\frac{3z'^2}{r'^5}\right) + 3\frac{x^2-y^2}{4}\frac{x'^2-y'^2}{r'^5}$$
$$+ 3xy\,(x'y'/r'^5) + 3xz\,(x'z'/r'^5) + 3yz\,(y'z'/r'^5) \quad\text{......(5)},$$

$$\frac{1}{m''}R' = \frac{r^2-3z^2}{4}\left(\frac{1}{D^3}-\frac{3\zeta^2}{D^5}\right) + 3\frac{x^2-y^2}{4}\frac{\xi^2-\eta^2}{D^5}$$
$$+ 3xy\,(\xi\eta/D^5) + 3xz\,(\xi\zeta/D^5) + 3yz\,(\eta\zeta/D^5) \quad\text{......(5')},$$

in which it will be noticed that the coordinates of the Moon are separated from those of the Sun or of the Planet. It is now necessary to express R, R' by means of the polar coordinates of the planet and of the Earth (or of G), referred to the Sun, and those of the Moon referred to the Earth.

The notation of Chap. IX. will be used whenever it differs from that of

previous chapters. We suppose the ecliptic to be a fixed plane perpendicular to the axis of z. As before,

L = Distance of the Moon from its node,
h = Longitude of the node,
γ = Sine of half the inclination of the lunar orbit.

We have then (fig. 5, Art. 73)

$$x = r\cos L\cos h - r\cos i\sin L\sin h,$$
$$y = r\cos L\sin h + r\cos i\sin L\cos h,$$
$$z = r\sin L\sin i;$$

or

$$\left.\begin{aligned} x &= (1-\gamma^2)\,r\cos(L+h) + \gamma^2 r\cos(L-h),\\ y &= (1-\gamma^2)\,r\sin(L+h) - \gamma^2 r\sin(L-h),\\ z &= 2\gamma\sqrt{1-\gamma^2}\,r\sin L \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Let V' = Longitude of the Earth as seen from the Sun. The longitude of the Sun, as seen from the Earth, will be $V'+180°$, and therefore

$$x' = -r'\cos V',\qquad y' = -r'\sin V',\qquad z' = 0\ldots\ldots\ldots\ldots\ldots(7).$$

The coordinates ξ, η, ζ, being those of P relative to E, are those of P relative to the Sun added to those of the Sun relative to E. The coordinates of P relative to the Sun may be deduced from (6) if we put

γ'' = Sine of half the inclination of the orbit of P to the ecliptic,
h'' = Longitude of its node on the ecliptic,
V'' = Longitude of P as seen from the Sun, reckoned along the ecliptic to its node and then along its orbit,
r'' = Solar radius vector of P,

for γ, h, $L+h$, r, respectively; the coordinates of the Sun are given by (7). We therefore obtain

$$\left.\begin{aligned} \xi &= -r'\cos V' + (1-\gamma''^2)\,r''\cos V'' + \gamma''^2 r''\cos(V''-2h''),\\ \eta &= -r'\sin V' + (1-\gamma''^2)\,r''\sin V'' - \gamma''^2 r''\sin(V''-2h''),\\ \zeta &= 2\gamma''\sqrt{1-\gamma''^2}\,r''\sin(V''-h'') \end{aligned}\right\}\ldots\ldots(8).$$

Whence

$$D^2 = \xi^2+\eta^2+\zeta^2$$
$$= D_0^2 + 4\gamma''^2 r'r''\sin(V'-h'')\sin(V''-h''),$$

where

$$D_0^2 = r'^2 + r''^2 - 2r'r''\cos(V'-V'')\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

It will be unnecessary to consider powers of γ'' beyond the second; we shall therefore have

$$\frac{1}{D^q} = \frac{1}{D_0^q} + \frac{q\gamma''^2 r'r''}{D_0^{q+2}}\{\cos(V'+V''-2h'') - \cos(V'-V'')\}\ldots\ldots(10).$$

By means of the formulæ (6)—(10), R, R' can be expressed in terms of r, r', r'', L, V', V'', h, h', h'', γ, γ''.

310. *Development of the Disturbing Functions.*

We shall first expand the expressions obtained for the various parts of the disturbing functions, by substituting elliptic values for the coordinates of the three bodies, and then show how non-elliptic terms present in these coordinates may be taken into account. When this has been done, the disturbing functions are to be expressed as sums of periodic terms.

(*a*) *The portions which depend only on the coordinates of the Moon.*

Neglecting powers of γ beyond the fourth, we obtain, from (6),

$$\tfrac{1}{4}(r^2 - 3z^2)/r^2 = \tfrac{1}{4}(1 - 6\gamma^2 + 6\gamma^4) + \tfrac{3}{2}\gamma^2(1 - \gamma^2)\cos 2L,$$

$$\tfrac{3}{4}(x^2 - y^2)/r^2 = \tfrac{3}{4}(1 - \gamma^2)^2 \cos(2L + 2h) + \tfrac{3}{4}\gamma^4 \cos(2L - 2h) + \tfrac{3}{2}\gamma^2(1 - \gamma^2)\cos 2h;$$

corresponding expressions may be obtained for $\tfrac{3}{2}xy/r^2$, $\tfrac{3}{2}xz/r^2$, $\tfrac{3}{2}yz/r^2$.

As in Chap. IX., let g be the distance of the lunar perigee from the node and f the true anomaly. Then

$$L = g + f,$$

and, from the expressions given in Art. 39,

$$r^2/a^2 = 1 + \tfrac{3}{2}e^2 - (2e - \tfrac{1}{4}e^3)\sin l - \dots,$$

$$\frac{r^2}{a^2}\cos(2f + \alpha) = (1 - \tfrac{5}{2}e^2)\cos(2l + \alpha) + e\cos(3l + \alpha) - 3e\cos(l + \alpha) + \dots,$$

where α may be any angle.

By giving to α suitable values, all the five functions $\tfrac{1}{4}(r^2 - 3z^2)$, $\tfrac{3}{4}(x^2 - y^2)$, etc., present in R, R', can be expressed in series of cosines or sines involving l, g, h in their arguments and e, γ in their coefficients. Moreover, the orders of the coefficients can be associated with the multiples of l, g, h in the corresponding arguments, by the rules obtained in Chap. VI. Putting M_0 for $l + g + h$, it is easily seen that

$$r^2 - 3z^2 = \Sigma A_0 e^k \cos kl + \gamma^2 \Sigma B_0 e^k \cos(2M_0 - 2h \pm kl),$$

where k is a positive integer and where A_0, B_0 are coefficients of zero order containing powers and products of e^2, γ^2. The other four functions depending only on the coordinates of the Moon may be similarly treated.

(*b*) *The parts which involve the coordinates of the Sun and of the Planet in the second degree.*

By means of the formulæ (8), we find the values of ζ^2, $\xi^2 - \eta^2$, $\xi\eta$, $\xi\zeta$, $\eta\zeta$ expressed as sums of cosines or sines. The arguments of these terms will contain multiples of V', V'', h'', and the coefficients will contain γ'' and will

have r'^2, r''^2 or $r'r''$ as factors. They are expanded in terms of the elliptic elements of the Sun and of the planet by the formulæ

$$\left.\begin{aligned} r'/a' &= 1 + \tfrac{1}{2}e'^2 - e'\cos l' - \ldots, \qquad & V' &= M_0' + 2e'\sin l' + \ldots, \\ r''/a'' &= 1 + \tfrac{1}{2}e''^2 - e''\cos l'' - \ldots, \qquad & V'' &= M_0'' + 2e''\sin l'' + \ldots \end{aligned}\right\} \ldots(11),$$

where M_0', M_0'' denote the mean longitudes of the Earth and the Planet respectively. The portions of R which depend only on the coordinates of the Sun, present no difficulty.

In the same manner as with the coordinates of the Moon, the composition of the argument of any term in the development of these expressions may be associated with the order of its coefficient, though the connection is by no means so simple. For instance, it may be shown that

$$\xi^2 - \eta^2 = \Sigma A_0 \gamma''^{2i} e'^{k'} e''^{k''} \cos\{2M_0'' - 2iM_0'' + j(M_0' - M_0'') + 2ih'' \pm k'l' \pm k''l''\},$$

where A_0 is a coefficient of zero order: k', k'' are positive integers or zeros, and i, j have positive integral or zero values such that $i + j < 3$.

(c) *The parts of R' arising from the Divisors D^q.*

The equation (10) shows that it is only necessary to consider the divisors D_0^q. They are the functions which cause the great difficulty in finding the planetary inequalities in the Moon's motion; the difficulty is of the same nature as that encountered in the planetary theory and it arises from the near equality of r', r'' or of n', n'' in the cases of those planets which are not far from the Earth (see Art. 9).

We can expand $1/D_0^q$, by means of Legendre's coefficients, in the form

$$D_0^{-q} = \tfrac{1}{2}B_q^{(0)} + B_q^{(1)}\cos(V' - V'') + B_q^{(2)}\cos 2(V' - V'') + \ldots,$$

where $B_q^{(j)}$ is a homogeneous function of r', r''; when r', r'' are comparable with one another in magnitude, these coefficients diminish very slowly and it becomes frequently necessary to consider terms in which j is a large number*. In the case of a superior planet, expansion must be made in powers of r'/r'' and, in the case of an inferior planet, in powers of r''/r'.

Substituting the values of r', r'', V', V'', given by (11), it is easily seen that

$$D_0^{-q} = \Sigma A_0 e'^{k'} e''^{k''} \cos\{j(M_0' - M_0'') \pm k'l' \pm k''l''\},$$

in which A_0 is a homogeneous function of a', a'' and of zero order with respect to e', e'', and j, k', k'' have positive integral or zero values.

* Radau's method (*Recherches*, pp. 17—31), for abbreviating the calculations in such cases, should be consulted.

(d) The terms arising in R from non-elliptic terms present in the coordinates of the Earth, the Planet and the Moon.

Two methods may be used for these terms. We may either consider r', v' (and also U', if the terms dependent on the motion of the ecliptic be not neglected) to receive small increments $\delta r'$, $\delta v'$ and then expand the formula (3) of Art. 108 in powers of these increments by means of Taylor's theorem, substituting for r', v' their elliptic values and for $\delta r'$, $\delta v'$ the small terms given by the planetary theory. Or we may suppose the additional terms to be given as small corrections to the elements of the solar orbit, in which case the development (5) of Art. 114 will be available after the changes of notation, necessary to express the result in Delaunay's form (Arts. 123, 180), have been made. The same methods may also be employed to take into account any non-elliptic terms present in the coordinates of the planet.

The solar terms present in the coordinates of the Moon cannot, in all cases, be neglected. In the process of eliminating, by Delaunay's method, the periodic terms of R which are due to the action of the Sun, the lunar elements present in R' will be changed at each operation. Instead of inserting the changes, thus produced, by adding them to the elements, it will generally be more convenient to suppose that the elliptic values of the coordinates receive small increments, these increments being the principal solar terms which occur in the unreduced values of the coordinates, as given by Delaunay. Numerical values may usually be substituted in all parts of the coefficients of the new terms, except in the characteristics.

311. After the various processes, outlined above, have been carried out, it is only necessary to multiply the series obtained for the various parts of R or R' and to express them as sums of cosines of angles. To do this in any general manner, would involve enormous labour due chiefly to the divisors D^q; and much of the labour would be without result, because the great majority of the terms have quite insensible coefficients in the coordinates of the Moon. The plan usually adopted consists in trying to discover the terms which have long periods and which, in consequence, may have coefficients large in comparison with their order when the equations of motion are integrated. Certain short-period terms which are either associated with these long-period terms, or which have an independent existence in the disturbing function, must also be included when there is a possibility of a large coefficient in one of the coordinates. In every case, the methods by which R, R' have been developed, give the order of the coefficient in the disturbing function in relation to the eccentricities and inclinations. No further rules can be given to guide us in the choice of these terms. Many of them have been indicated by observation: others have been obtained directly from theory in the course of investigations into the effects of planetary action.

The method here outlined for the treatment of the disturbing function was first given by Dr Hill* and was afterwards extended and applied to many planetary inequalities by M. Radau†. Combined with Hill's method of integration (Arts. 304—307), it forms the only complete and generally effective method known up to the present time for the investigation of the planetary inequalities.

There are several ways in which the calculation of the coefficient of a term in R', with a given argument, may be abridged; to give an account of them would lead us outside the limits of this treatise. The reader is referred to Radau's memoir and also to Tisserand's *Mécanique Céleste*‡ which contains an account of this memoir.

We shall illustrate the methods of the previous articles by applying them to the calculation of two celebrated inequalities—one due to the direct action of Venus and the other due to its indirect action.

312. *Example of an inequality due to the Direct action of Venus.*

There is a term in R' of period $2\pi/(l_0 + 16n' - 18n'')$, where n'' is the mean motion of Venus. Observation furnishes, for the daily motions,

$$l_0 = 47033''\cdot97, \quad n' = 3548''\cdot19, \quad n'' = 5767''\cdot67.$$

The daily motion of this inequality is therefore $-13''\cdot0$, giving a period of 273 years.

It can be shown§ that the term in R', having this period, is

$$-0''\cdot00133 n'^2 a^2 e \cos(l + 16M_0' - 18M_0'' + 2h'').$$

We therefore have, on applying the formulæ of Art. 306,

$$p = -273, \quad A' = -0''\cdot00133 pe = 0''\cdot0199;$$

$$i = 1, \quad i' = 0 = i''; \quad j = 2, \quad j' = 1, \quad j'' = 0;$$

and the equations (4) give

$$\delta M_0 = 16''\cdot6 \sin(l + 16M_0' - 18M_0'' + 2h''),$$

which also gives the approximate value of the term in longitude, since δl is nearly equal to δM_0 and since the equations for δa, δe, $\delta\gamma$, δh only give small coefficients. The more accurate value of the coefficient, when terms of higher orders are included, is $14''\cdot4$‖.

This is the largest known periodic inequality in longitude, produced by the action of the planets. Indeed, according to the table given by Radau at the end of his memoir, no other inequality in longitude has a coefficient so great as $1''$, although there are several greater than half a second; the majority of the inequalities are of comparatively short period—either approximating to the lunar month or having a period of a few years.

* "On certain Possible Abbreviations in the Computation of the Long-Period Inequalities of the Moon's Motion due to the Direct Action of the Planets," *Amer. Journ. Math.* Vol. VI. pp. 115—130.

† *Recherches* etc.

‡ Vol. III. Chap. XVIII.

§ Tisserand, *Méc. Cél.* Vol. III. p. 396.

‖ Radau, *Recherches* etc., p. 64.

The inequality just calculated was discovered by Hansen*, who found by theory a coefficient of 27″·4. He also noticed another inequality with a mean motion $8n''-13n'$ and a coefficient 23″·2. In both cases Hansen was in error; the former coefficient has just been seen to be about 14″·4, while Delaunay† and others have shown that the coefficient of the latter term is less than 0″·004. The values of the coefficients, which Hansen obtained by a discussion of the observations and which he adopted in his tables, were 15″·34 and 21″·47, respectively, including the parts due to the indirect action (see Art. 313).

The Indirect Action of a Planet.

313. A very simple formula can be obtained for this in many cases. Neglecting the perturbations of the plane of the ecliptic and the ratio of the parallaxes, we have, by Art. 116,

$$\delta R = -(3R/r')\,\delta r' - (\partial R/\partial v)\,\delta V'.$$

Let us confine our attention to the term $m'r^2/4r'^3$ of R (Art. 108), since most of the larger inequalities of long period will arise from this term. Substituting in the expression for δR and neglecting the solar eccentricity, we obtain immediately

$$\delta R = -(\tfrac{3}{4}\,n'^2a^2/a')\,\delta r'.$$

Suppose that the solar tables give an inequality $\delta r' = a'A\cos\theta$, where A, θ are independent of the lunar elements. We obtain

$$\delta R = -\tfrac{3}{4}\,n'^2a^2A\cos\theta.$$

Using the equations (4), we have i, i', i'', j', j'' zero and $j = 2$. Whence

$$\delta M_0 = \delta l = \tfrac{3}{2}\times 0{\cdot}149\,pA\sin\theta = \tfrac{2}{9}\,pA\sin\theta,$$

approximately.

If we suppose further that the inequality $\delta r'$ is of long period and that it arises principally from a variation $\delta a'$ of a', a direct approximate relation between δv and $\delta V'$ can be deduced. For (Art. 81)

$$r' = a'(1 - e'\cos l' + \dots),\qquad V' = \int n'dt + \epsilon_1' + 2e'\sin l' + \dots,$$

and therefore, owing to the various conditions assumed above,

$$\delta r' = \delta a',\qquad \delta V' = \int \delta n'dt.$$

But since $n'^2a'^3 = m'$, we have

$$2a'\delta n' = -3n'\delta a' = -3n'\delta r' = -3n'a'A\cos\theta.$$

Therefore $$\delta V' = -\tfrac{3}{2}\,\frac{n'A}{M}\sin\theta = -\tfrac{3}{2}\,pA\sin\theta.$$

* "Auszug aus einem Briefe," etc. *Astr. Nach.* Vol. xxv. Cols. 325—332; "Lettre à M. Arago," *Comptes Rendus*, Vol. xxiv. pp. 795—799.

† "Sur l'Inégalité lunaire à longue periode due à l'action perturbatrice de Vénus et dépendant de l'argument $13l' - 8l''$," *Conn. des Temps*, 1863, Additions, pp. 1—56. The result is given on p. 46.

On combining this result with the value just obtained for δM_0, we find

$$\delta v = \delta M_0 = -\tfrac{4}{27}\,\delta V' = -\tfrac{1}{7}\,\delta V',$$

approximately. In this case we can therefore obtain an approximate idea of the magnitude of the coefficient in longitude, by dividing the corresponding inequality in the Earth's longitude by -7.

For example, the solar tables* give an inequality of period

$$2\pi/(13n' - 8n'') = 239 \text{ years},$$

due to the action of Venus. In longitude, this is

$$\delta V' = +1''{\cdot}92 \sin(13M_0' - 8M_0'' + 132^\circ).$$

Multiplying the coefficient by $-4/27$, we obtain for the corresponding inequality in the Moon's motion, due to the indirect action of Venus,

$$\delta v = -0''{\cdot}284 \sin(13M_0' - 8M_0'' + 132^\circ);$$

the correct value, as found by Delaunay †, being

$$\delta v = -0''{\cdot}272 \sin(13M_0' - 8M_0'' + 138^\circ).$$

The inequality having this period, due to the indirect action of Venus, is therefore much greater than that, with the same period, produced by the direct action (Art. 312).

For a complete investigation of the inequalities produced by the direct and indirect actions of the planets, the reader is referred to Radau's memoir. A large number of references to the labours of other investigators on the same subject is also given. To these may be added an important paper by Newcomb ‡, "Theory of the Inequalities in the Motion of the Moon produced by the Action of the Planets," in which the whole theory of the subject is treated in a very general manner.

Inequalities arising from the Figure of the Earth.

314. Let A, B, C be the moments of inertia of the Earth about three rectangular axes meeting in the centre of mass, and let I be the moment of inertia about the line connecting this point with the centre of mass of the Moon. The difference of the attractions on the Moon, of the Earth and of a spherical body of equal mass, produces a potential §

$$(A + B + C - 3I)/2r^3.$$

We suppose that one principal axis of the Earth is its polar axis and that the moments of inertia about the other two axes are equal. Let $B = A$, and

* *Ann. de l'Obs. de Paris* (*Mém.*), Vol. IV. p. 35. The inequality is given in the form

$$-1''{\cdot}283 \sin(13M_0' - 8M_0'') + 1''{\cdot}425 \cos(13M_0' - 8M_0'').$$

† See footnote, p. 259.

‡ *Astron. Papers for Amer. Eph.* Vol. v. Pt. III. pp. 97–295.

§ E. J. Routh, *Rigid Dynamics*, Vol. II. Art. 481.

let d be the declination of the Moon. We then have

$$I = A\cos^2 \mathrm{d} + C\sin^2 \mathrm{d};$$

and the new part to be added to the disturbing function is

$$\delta R = \frac{1}{2r^3}(2A + C - 3A\cos^2 \mathrm{d} - 3C\sin^2 \mathrm{d})$$

$$= \frac{C-A}{2r^3}(1 - 3\sin^2 \mathrm{d}) = \frac{\mu k'}{r^3}(\tfrac{1}{3} - \sin^2 \mathrm{d}) \text{ .... } (12);$$

where μ is the sum of the masses of the Earth and the Moon, and

$$\mu k' = \tfrac{3}{2}(C - A).$$

It is proved in works on the figure of the Earth * that, if the Earth's surface be supposed to be an equi-potential surface,

$$\mu k' = E\textsc{r}^2(\alpha - \tfrac{1}{2}\beta),$$

where E is the Earth's mass, R its equatoreal radius, α its ellipticity and β the ratio of the centrifugal force to gravity at the equator.

The numerical determination of $\mu k'$ may be made in several ways. It is possible to find it by the reverse process of comparing the theoretical values of the coefficients of the principal terms produced by the figure of the Earth on the motion of the Moon, with those deduced from observation; owing to the near equality of the periods of these terms with the periods of certain terms produced by planetary action—terms whose coefficients are not known with a great degree of certainty—this method is not capable of very great accuracy. The value may be deduced from the latter of the formulæ given above for $\mu k'$, by obtaining α from geodetic measures and β from pendulum observations; this method involves an assumption concerning the interior constitution of the globe. Thirdly, it may be obtained from the first formula—and Hill so finds it†—by a discussion of pendulum observations, made to find the intensity of gravity at various stations on the surface of the Earth. A determination has also been made by comparing the observed and the calculated values of the yearly precession of the Equinoxes.

315. Let x (Fig. 5, Art. 73) be the ascending node of the Ecliptic on the Equator—the place from which longitudes are reckoned—and let ω_1 be the inclination of these two planes. If z' be the pole of the equatoreal plane, we have $\quad zz' = \omega_1, \quad z'M = 90^\circ - \mathrm{d}, \quad zM = 90^\circ - U, \quad z'zM = 90^\circ - v.$

The triangle $z'zM$ therefore gives

$$\sin \mathrm{d} = \sin U \cos \omega_1 + \cos U \sin \omega_1 \sin v.$$

We also have $$\sin U = \sin i \sin L,$$

$$\cos U \cos(v - h) = \cos L, \qquad \cos U \sin(v - h) = \sin L \cos i;$$

* e.g. J. H. Pratt, Art. 85.

† In chapter v. of his Memoir "Determination of the Inequalities of the Moon's Motion which are produced by the Figure of the Earth: a supplement to Delaunay's Lunar Theory," *Astron. Papers for Amer. Eph.* Vol. III. pp. 201—344.

and therefore

$$\cos U \sin v = \cos^2 \tfrac{1}{2}i \sin (L+h) - \sin^2 \tfrac{1}{2}i \sin (L-h).$$

Hence

$$\sin \mathrm{d} = \cos \omega_1 \sin i \sin L + \cos^2 \tfrac{1}{2}i \sin \omega_1 \sin (L+h) - \sin^2 \tfrac{1}{2}i \sin \omega_1 \sin (L-h).$$

Putting $\sin \frac{1}{2}i = \gamma$ and neglecting powers of γ beyond the first, we obtain

$$\sin \mathrm{d} = \sin \omega_1 \sin (L+h) + 2\gamma \cos \omega_1 \sin L;$$

$$\begin{aligned}\delta R &= \mu k' (\tfrac{1}{3} - \sin^2 \mathrm{d})/r^3 \\ &= (\mu k'/r^3)[\tfrac{1}{3} - \tfrac{1}{2}\sin^2 \omega_1 + \tfrac{1}{2}\sin^2\omega_1 \cos 2(L+h) - \gamma \sin 2\omega_1 \{\cos h - \cos (2L+h)\}].\end{aligned}$$

As the method of integration will be that of Arts. 304—307, it is necessary to substitute elliptic values for r and L. We put

$$r = a(1 - e\cos l + \ldots), \quad L = g + l + 2e \sin l + \ldots.$$

The terms in δR which will give the greatest coefficients are those of long period: it is easily seen that, after the substitution of elliptic values, there is only one such term—that with argument h. We therefore take

$$\delta R = -\mu k' (\gamma/a^3) \sin 2\omega_1 \cos h.$$

All that now remains is the application of the formulæ (4). Since the diurnal motion of the node is $-190''\cdot 77$, we find

$$p = n'/h_0 = -(3548''\cdot 2 \div 190''\cdot 77) = -18\cdot 60;$$

$$i = i' = 0, \quad i'' = 1; \quad j = -3, \quad j' = 0, \quad j'' = 1;$$

$$A' = -p \frac{\mu}{a^3} \frac{k'\gamma}{n'^2 a^2} \sin 2\omega_1 = 18\cdot 60 \times \frac{n^2}{n'^2} \frac{k'}{a^2} \gamma \sin 2\omega_1.$$

Substituting and retaining only three places of decimals in the coefficient of $A' \cos h$, we obtain

$$\left.\begin{array}{lll} \delta a = 0, & \delta e = 0, & \delta\gamma = -0\cdot 414 A' \cos h, \\ \delta M_0 = +0\cdot 690 A' \cos h, & \delta l = +1\cdot 118 A' \sin h, & \delta h = +9\cdot 286 A' \sin h \end{array}\right\} \ldots (13).$$

Hill finds, by his discussion of pendulum observations *,

$$(k'/a^2) \sin 2\omega_1 = 0''\cdot 072854.$$

Hence, with the values of γ, n'/n given in Art. 306, $A' = 10''\cdot 99$. The results (13) then give

$$\delta\gamma = -4''\cdot 55 \cos h,$$

$$\delta M_0 = +7''\cdot 58 \sin h, \qquad \delta l = +12''\cdot 28 \sin h, \qquad \delta h = +102''\cdot 0 \sin h.$$

* On p. 340 of the memoir referred to in Art. 314 above. The quantity here called $k' \sin 2\omega_1$ is denoted by β_2.

316. To find the corresponding inequalities in the coordinates, it is only necessary to subject Delaunay's results for the elliptic and solar terms to a variation δ and to insert the above values. It is sufficient for our purposes to take, in longitude,

$$v = M_0 + 2e \sin l - \gamma^2 \sin 2(g + l),$$

$$\delta v = \delta M_0 + 2e\delta l \cos l - 2\gamma\delta\gamma \sin 2(g + l) - 2\gamma^2 (\delta g + \delta l) \cos 2(g + l).$$

When the values of δl, $\delta\gamma$, $\delta g + \delta l = \delta M_0 - \delta h$, are substituted, it will be found that the first term only gives an inequality so great as $1''$. Hence

$$\delta v = \delta M_0 = +7''\cdot 58 \sin h.$$

In latitude, we have

$$\sin U = \sin i \sin (g + l).$$

Putting $\gamma = \sin \frac{1}{2} i$ and neglecting quantities of the order γ^3, we find

$$\begin{aligned} \delta U &= \quad 2\delta\gamma \sin (g + l) + 2\gamma (\delta M_0 - \delta h) \cos (g + l) \\ &= -9''\cdot 10 \sin (g + l) \cos h - 8''\cdot 48 \cos (g + l) \sin h \\ &= -8''\cdot 79 \sin (h + g + l) - 0''\cdot 31 \sin (g + l - h). \end{aligned}$$

The only inequalities, having coefficients greater than $1''$ in longitude and latitude, have therefore arguments equal to the longitude of the node and to the mean longitude, respectively; the periods are 18·6 years and one mean sidereal month. The coefficients, as found by Hill* who followed Delaunay's method exactly, are $+7''\cdot 67$ and $-8''\cdot 73$, so that the calculations made above give the values with considerable accuracy. The extensions necessary to find the coefficients of the other periodic terms can be easily made by the method used here: there are several of about half a second of arc in magnitude.

Other determinations of the inequalities due to the figure of the Earth are to be found in the works of Laplace†, de Pontécoulant‡, Hansen§, Tisserand‖, etc.

The otion of the Ecliptic.

317. Owing to planetary action, the plane of the Earth's orbit, which has been hitherto considered to be the plane of reference, is not fixed. If the plane of reference, e.g. the ecliptic at a given date, had been fixed, this motion of the ecliptic, being very small, would have produced but little effect on the motion of the Moon when introduced into R. But it is usual to use the instantaneous ecliptic as the plane for the measurement of longitudes and

* Mem. cit. pp. 341, 342.

† *Méc. Cél.* Pt. II. Book VII. Chap. II.; Book XVI. Chap. III.

‡ *Sys. du Monde*, Vol. IV. Chap. IV.

§ *Darlegung*, I. pp. 459—474, II. pp. 273—322.

‖ *Méc. Cél.* Vol. III. pp. 144—149, 155—160.

latitudes. Hence the apparent place of the Moon will be affected to an extent which is comparable with the motion of the ecliptic. Since the inclination of the lunar orbit is a small quantity, and since the line of intersection of two consecutive positions of the ecliptic has a motion so small that it may be neglected in comparison with the rotation of the ecliptic, the principal effect produced by referring the Moon's place to the moving ecliptic will occur in the latitude of the Moon. The approximate fixture of the node of the ecliptic reduces the consideration of its motion to that of a small rotation about a fixed line.

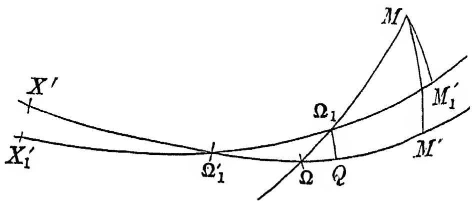

Fig. 11.

318. Let $\Omega\Omega_1 M$ be the position of the lunar orbit, and let $X'\Omega_1'M'$, $X_1'\Omega_1'M_1'$ be the two ecliptics at times $t, t+dt$. The position of the Moon will have changed during the interval dt; but since two small changes may be calculated by considering their effects separately and adding the results, we can consider the position of the Moon as unchanged in finding the apparent change in time dt due to the motion of the ecliptic. Let the rate of rotation of the ecliptic be denoted by $\beta n'$.

Draw $\Omega_1 Q$ and MM' perpendicular to $X'M'$. As before, we put

$$X'\Omega = h, \quad \Omega M = L, \quad X'M' = v, \quad M'M = U;$$

and further

$$X'\Omega_1' = h' = X_1'\Omega_1',$$

since X', X_1' are now departure points and Ω_1' is fixed with reference to them.

We have

$$dh = \Omega Q = \Omega_1 Q \cot i = \beta n' dt \cot i \sin(h - h').$$

Also, by considering a point on the ecliptic 90° in advance of Ω, we obtain

$$-di = \beta n' \delta t \sin(90° + h - h') = \beta n' dt \cos(h - h').$$

The equations for h, i are therefore

$$\frac{dh}{dt} = \beta n' \cot i \sin(h - h'), \qquad \frac{di}{dt} = -\beta n' \cos(h - h').$$

When the motion of the ecliptic is neglected, $h = h_0 t + \text{const.}$ Since β is a very small coefficient, we may integrate the equations on the supposition

that h has this value and that i is constant in their right-hand members. The new parts of h, i are therefore given by

$$\delta h = -(\beta n'/h_0) \cot i \cos (h - h'), \qquad \delta i = -(\beta n'/h_0) \sin (h - h').$$

Further, $$\delta (g + l) = -\Omega \Omega_1 = -\frac{\delta h}{\cos i} = \frac{\beta n'}{h_0} \frac{\cos (h - h')}{\sin i}.$$

The latitude is given by the equation, $\sin U = \sin i \sin (g + l)$. Hence

$$\begin{aligned} \delta U \cos U &= \delta i \cos i \sin (g + l) + \delta (g + l) \sin i \cos (g + l) \\ &= (\beta n'/h_0) \{-\cos i \sin (g + l) \sin (h - h') + \cos (g + l) \cos (h - h')\}, \end{aligned}$$

which, by considering the triangles $M\Omega\Omega_1'$, $M\Omega_1'M_1'$, becomes

$$\delta U = (\beta n'/h_0) \cos (v - h').$$

The period of the inequality is therefore the same as that of the Moon. The annual motion of the ecliptic is $0''\cdot 48$, and the node of the Moon's orbit makes a revolution in 18·6 years. Hence

$$\beta n' = 0''\cdot 48, \quad 2\pi/h_0 = 18\cdot 6;$$

therefore $$\beta n'/h_0 = 0''\cdot 48 \times 18\cdot 6 \div 6\cdot 18 = 1''\cdot 42,$$

giving $$\delta U = 1''\cdot 42 \cos (v - h').$$

The corresponding inequality in longitude is much smaller. Its period is that of the mean motion of the node and its coefficient is less than one-third of a second. The calculation of it presents some difficulties and requires a more extended investigation.

The above method of investigation was given by Adams in a "Note on the Inequality in the Moon's Latitude which is due to the secular change of the Plane of the Ecliptic *." A more complete investigation by Hill will be found in a paper "On the Lunar Inequalities produced by the Motion of the Ecliptic †." Reference may also be made to Hansen ‡, Radau §, Tisserand ‖.

The Secular Acceleration of the oon's ean otion.

319. The action of the planets produces a slow variation in the eccentricity of the Earth's orbit which is usually expressed in the form,

$$e' = e_0' - at + a't^2 + \dots.$$

The coefficients a, a', ... are quite insensible in the motion of the Earth,

* *M. N. R. A. S.* Vol XLI. pp. 385—403. *Coll. Works*, pp. 231—252. Godfray's *Lunar Theory*, Art. 113.

† *Annals of Math.* (U. S. A.), Vol. I. pp. 5—10, 25—31, 52—58.

‡ *Darlegung*, I. pp. 118—120, 490—491.

§ "Influence du Déplacement séculaire de l'Ecliptique," *Bull. Astron.* Vol. IX. pp. 363—373.

‖ *Méc. Cél.* Vol. III. pp. 136—140, 160—164.

but the first of them produces in the longitude of the Moon an effect which is easily noticeable when observations, extending over a hundred years or more, are discussed. When this expression, instead of e', is introduced into the disturbing function, it is evident that terms of the forms $t^j \cos(at+b)$ will be introduced into R and therefore into the coordinates of the Moon.

In order to make the investigation as brief as possible, we shall make two assumptions which have been verified by actual calculation. The first is that the only sensible coefficient in the expression for e' is α and that the squares and higher powers of α may be neglected; the second is that terms of the form $t \sin(at+b)$, which will arise in the final expressions for the coordinates, have coefficients so small that they may be neglected. It is required, therefore, to find what non-periodic terms are produced in the coordinates by the term $-\alpha t$. The method of Chapter VII. will be used for the investigation.

320. Let us consider how non-periodic terms were produced in the right-hand member of the equation (1), Art. 130. It was seen in Chap. VII. that the first approximation to the coefficient of any term in δu was obtained by simply considering the corresponding term in δR: terms of a lower characteristic could be neglected. As only non-periodic terms are required here, and as it was shown in Art. 116 that $d'R/dt$ contained no such terms, we have, from equation (2), Art. 130, neglecting a/a',

$$P = r\,\partial R/\partial r + \text{const.} = 2R + \text{const.}$$

There is only one non-periodic term in R which need be considered, namely, that containing e'^2. We have therefore, from Art. 114,

$$P = \tfrac{3}{4}m^2 e'^2/a + \text{const.}$$

Equation (1), Art. 130, then becomes, since e, γ are neglected,

$$-a^3\frac{d^2}{dt^2}\delta u - \delta u = \tfrac{3}{4}\frac{m^2}{a}e'^2 + \text{const.}$$

Putting $e' = e_0' - \alpha t$ and neglecting squares of α, this equation furnishes

$$a\delta u = -\tfrac{3}{4}m^2 e'^2 + \text{const.}, \quad \text{or}, \quad a\delta u = \tfrac{3}{4}m^2(e_0'^2 - e'^2) + \text{const.},$$

giving the inequality produced in the parallax.

321. Next, consider the equation (5), Art. 131, for the longitude. Neglecting e, γ, a/a', we obtain

$$\frac{d}{dt}\delta v = \frac{\delta h}{a^2} + \frac{1}{a^2}\int\frac{\partial R}{\partial \xi}dt + \frac{2h_0}{a}\delta u.$$

The integral in the right-hand member gives no term free from sines or cosines. We therefore get, on substituting for δu, and putting $h_0 = na^2$,

$$\frac{d}{dt}\delta v = \frac{\delta h}{a^2} + \text{const.} + \tfrac{3}{2} m^2 n (e_0'^2 - e'^2).$$

Now n was defined so that all constant terms in this equation should vanish: this definition will be retained. As $e_0'^2 - e'^2$ contains t as a factor, the term involving this quantity cannot be made to vanish. Hence, by a suitable determination of δh, we find, on integration,

$$\delta v = \tfrac{3}{2} m^2 n \int_0^t (e_0'^2 - e'^2)\, dt.$$

The additive constant is put zero, according to the remarks of Art. 158.

The general expression for the longitude therefore becomes

$$v = nt + \epsilon + \tfrac{3}{2} m^2 n \int_0^t (e_0'^2 - e'^2)\, dt + \text{periodic terms}$$
$$= nt + \epsilon + \tfrac{3}{2} mn' e_0' \alpha t^2 + \text{periodic terms.}$$

Let the unit of time be one Julian year; n' will then be the angle described by the Sun in one year. The planetary theory gives, for the epoch 1850·0*,

$$e_0' = 0{\cdot}016771, \quad \alpha = 0{\cdot}0000004245, \quad n' = 1295977'', \quad m = 0{\cdot}07480.$$

The term in v, involving t^2, is therefore†

$$+ 10''{\cdot}35\, (t/100)^2.$$

The presence of this term in the longitude is usually expressed by saying that the mean angular velocity of the Moon is not quite constant but has a secular acceleration of $10''{\cdot}35$ per century; the more correct statement being that the mean motion is increasing at the rate of $2 \times 10''{\cdot}35$ per century in a century.

322. This is approximately the value found by Laplace and it requires considerable modification when we proceed to higher powers of m and to the terms dependent on e^2, γ^2, etc. It is in the second and higher approximations that the difficulty of the subject arises. To obtain the next approximation, it is necessary to consider not only the non-periodic part of R but also those periodic terms which, in combination with periodic terms of equal arguments, may produce non-periodic terms in the longitude equation: and it is to be remembered, when integrating the equations of motion, that e' is variable. For instance, to get the next approximation in powers of m by this method,

* *Ann. de l'Obs. de Paris (Mém.)*, Vol. IV. p. 102.

† The result obtained for this term by Adams and Delaunay (see Art. 296) is $+10''{\cdot}66$, owing to the use of a slightly different value for α.

it is necessary to retain (i) the parts of R of characteristic zero, (ii) the non-periodic term of the order e'^2, (iii) the periodic terms of characteristic e', (iv) the values of r', v' as far as the order e'^2 in the non-periodic terms and to the order e' in the periodic terms. The details of the next approximation are too extended to be given here; they may be found, calculated to the order m^4 after this method, in a memoir by Cayley (see Art. 296 above). The value to the order m^5 is

$$-\left(\tfrac{3}{2}m^2 - \tfrac{3771}{64}m^4 - \tfrac{34047}{64}m^5\right) n\int (e_0'^2 - e'^2)\, dt.$$

The coefficients corresponding to the second and third terms are $-2''\cdot 27$, $-1''\cdot 54$, causing the acceleration to be diminished to $6''\cdot 5$. Delaunay * finds $6''\cdot 11$ to be the complete theoretical value.

It is necessary to make one further remark. The value of e' does not always continue to diminish; after a period of about 24000 years it will have reached its minimum value and begin to increase again, attaining a maximum after the lapse of another period of similar magnitude. Were it not for this fact, the period of revolution of the Moon would go on increasing and its distance diminishing until it was brought within the limits of the terrestrial atmosphere. The periodic nature of the variation of e' prevents this small inequality from producing any great change in the relations between the Moon and the Earth.

The observable effect on the mean distance is quite insensible. This will be readily understood when it is mentioned that the annual mean approach of the Moon to the Earth, due to this cause, is less than one inch; in 200 years, the mean distance will be only fourteen feet smaller than at the present time, corresponding to a change in the parallax of less than one twenty-thousandth of a second of arc.

Since the expressions of c, g contain terms dependent on e'^2, direct observations of the motions of the Perigee and the Node will show similar secular changes. The values of c, g contain the terms $-\frac{9}{8}m^2e'^2$, $+\frac{9}{8}m^2e'^2$. The first approximations to the secular accelerations of these motions will therefore be

$$-\tfrac{9}{8}m^2n\int (e_0'^2 - e'^2)dt, \quad +\tfrac{9}{8}m^2n\int (e_0'^2 - e'^2)dt,$$

respectively. The first of these is very much altered by the further approximations. Delaunay† finds $-40''\cdot 0$ and $+6''\cdot 8$ for their complete values.

For further references on the subject of the secular acceleration, see Art. 296.

* "Calcul de l'accélération séculaire du moyen mouvement de la lune," *Comptes Rendus*, Vol. XLVIII. pp. 817—827.

† "Calcul des variations séculaires des moyens mouvements du périgée et du nœud de l'orbite de la lune," *Comptes Rendus*, Vol. XLIX. pp. 309—314.

TABLES OF NOTATION

AND

INDICES

I. REFERENCE TABLE OF NOTATION.

The numbers following the symbols refer to the articles in which the symbols are first used. Brackets denote that the symbol is defined and used with that definition only, in the articles which accompany it. Symbols which are defined and used in one article only, or in two consecutive articles only, are not generally included. The new *symbols occurring only in articles* 308–313 *are also omitted. The letters used in the figures are not included.*

General Notation I.–VIII., XI., XII. (1–178, 242–302, 319–322).	Delaunay IX., XIII. (179–201, 303–318).	Hansen X. (202–241).
a 12, a' 19, a_{2i} 247, A' 22, A_i A_i' 247, (A 64–66), (A A' 111, 115), a 247, A 128. b_{2i} 134, b'_{2i} 166, B' 22, (B 135, 158), b$_i$ b$_i'$ 264, B 128. c 68, c_i 139, C 20, C' 22, c 257, c$_0$ 267, C 128. d' 12, D 18, D_t 20. e 32, e' 53, (e_0' 319–322), E 3, E 32, e 261, Eq. 39. f 32, f' 110, F 3, (F' 3–8), (F_1 F_1' 5–9), F' 18. g 68, g_i 147, g 279, g$_0$ 280. h 12, h_0 77, (H 94–98). i 44. J_i ( ) 37. K' 22, K_i K_i' 279, K 128. l 32, L L_1 44. m 114, m_1 124, m' 3, M 3, M_i 265, m 18. n 18 and 48, n' 19, N 263. (p_i 94–104), P 130, $\mathfrak{P}_1$ 16, $\mathfrak{P}$ 75. (q_i 94–104).	a_0 192, A 183, A' 306. B B_1 183. c C 183. D 200. e_0 192. F 200. g G g' 179, (G) 183, (g) 184, g_i g_s G_i G_c 186, G' 190. h H h' 179, (H) 183, (h) 184, h_i h_s H_i H_c 186, H' 190. i i' i'' i''' 180. j j' j'' 306. K 184. l L l' 179, l_i l_s L_i L_c 186, (l) 195, L 309. M 305, M_0 306. p 306.	a_0 208, a_0' 220. b 230. c_0 c_0' 224, c_j c_j c_j' 228, C 233. e_0 208, e_0' 220, $\bar{E}$ 208. $\bar{f}$ 208, $\bar{f}'$ 220, f_0 225. g 208, g' 224. h 204, h_0 208. i' 217. J 217, J_0 219. K 217, K_0 219. l_0 208. m_1 220. n_0 208, n_0' 220, N 217, N_0 219. p p' 217, P 219, $\mathfrak{P}$ 204, $\mathfrak{P}_0$ 225. q q' 217, Q 219. $\bar{r}$ 208, $\bar{r}'$ 220, r_0 225, R 204. s 241.

E, M, μ, m', r, r', v, $\textsc{u}$, s, S, n', a', e', x, y, z, x', y', z' retain the same meanings as in the first column.

n, a, e, (and γ) refer first to the instantaneous orbit. In Art. 200, they denote the arbitrary constants.

For change of notation, see Art. 179.

E, M, μ, m', r, r', Δ, $\textsc{u}$, S, $R^{(i)}$, R_i retain the same meanings as in the first column.

n, a, e, ϵ, ϖ, θ, i, l, f, $\textsc{e}$, $\textsc{l}$ refer here to the instantaneous orbit.

For change of notation, see Art. 204.

II. GENERAL SCHEME OF NOTATION.

The symbols connected with the lunar orbit refer to undisturbed elliptic motion in Chap. III. *and to the Instantaneous Ellipse in Chap.* V.*; their meanings in Chaps.* VII., VIII., *together with those of other symbols, will be found in Table III. Accented letters, in general, refer to the solar orbit.*

Symbols.	Significations.
$E,\ M,\ m'$	Masses of Earth, Moon and Sun.
μ	$E+M$. ($\mu=1$ in de Pontécoulant's theory.)
$r,\ r',\ \Delta$	Distances of Moon and Sun from Earth, and of Moon from Sun.
S	Cosine of angle between r, r'.
$x,\ y,\ z$	Coordinates of Moon referred to Fixed axes through Earth.
$X,\ Y,\ z$	Coordinates of Moon referred to Moving axes through Earth.
$v,\ \sigma$	$X+Y\sqrt{-1},\ X-Y\sqrt{-1}$.
u	$1/r$.
$v,\ v'$	Longitudes of Moon and Sun reckoned on Ecliptic.
U	Latitude of Moon above Ecliptic.
s	Tan U.
F	Force Function for Motion of Moon.
R	Disturbing Function arising from solar action.
$\mathfrak{P},\ \mathfrak{T},\ \mathfrak{Z}$	Solar Forces on Moon, along r, perpendicular to r in plane of orbit, and perpendicular to plane of orbit.
$\mathfrak{P}_1,\ \mathfrak{T}_1,\ \mathfrak{Z}_1$	Solar Forces on Moon, along the projection of r, perpendicular to projection of r in Ecliptic, and perpendicular to Ecliptic.
$n,\ n'$	Mean Motions of Moon and Sun.
$a,\ a'$	Defined by equations $\mu=n^2a^3,\ m'+\mu=n'^2a'^3$.
$e,\ e'$	Eccentricities of lunar and solar orbits.
$\epsilon,\ \epsilon'$	Longitudes of Epochs of Mean Motions.
$\varpi,\ \varpi'$	Longitudes of lunar and solar Perigees.
θ	Longitude of lunar Node.
i	Inclination of lunar orbit to Ecliptic.
$\gamma,\ \gamma_1$	Defined by equations $\gamma=\tan i,\ \gamma_1=\sin\frac{1}{2}i$.
$f,\ f'$	Lunar and solar True Anomalies.
$w,\ w'$	" " Mean "
E	" Eccentric Anomaly.
L	Angular distance of Moon from its Node.
l	Latus Rectum of Moon's orbit.
h	Rate of description of areas by Moon in Ecliptic.
ζ	Exp. $\{(n-n')\,t+\epsilon-\epsilon'\}\sqrt{-1}$.
D	$\zeta d/d\zeta$.

III. COMPARATIVE TABLE OF NOTATION.

In the first column is the notation used by de Pontécoulant, the second and third columns contain the corresponding symbols used by Delaunay and Hansen. The fourth column contains the final definitions; those in square brackets refer to the methods of Chap. XI.

De Pont.	Del.	Hansen*.	Definitions.
n	n	$n_0(1+y-2\eta)$	Observed Mean Motion.
cn	l_0	n_0	Mean Motion of Mean Anomaly.
$(1-c)\,n$	g_0+h_0	$n_0(y-2\eta)$	" " Perigee. [$\mathrm{c}=c/(1-m)$.]
$(1-g)\,n$	h_0	$-n_0(\alpha+\eta)$	" " Node. [$\mathrm{g}=g/(1-m)$.]
n'	n'	$n_0'+n_0y'$	Observed Mean Motion of Sun.
		n_0y'	Mean Motion of Solar Mean Anomaly.
ϕ	l	g	Arg. of Principal Ell. Term in Long.
ϕ'	l'	g'	" Annual Equation.
ξ	D	$g+\omega-g'-\omega'$	Half Arg. of Variation.
η	F	$g+\omega$	Arg. of Principal Term in Lat.
m	m		$m=n'/n,\quad m_1{}^2=n'^2_0/n_0{}^2(1+\mu/m')$. [$\mathrm{m}=n'/(n-n')$.]
a	a	a_0	$a^3n^2=\mu=a_0{}^3n_0{}^2$. [For a, see Arts. 255, 273.]
a'	a'	a_0'	$a'^3n'^2=m',\quad a_0'^3n_0'^2=m'+\mu$.
e†	e		Ecc., defined by Principal Ell. Term in Long. [For e, see Arts. 261, 274.]
		e_0	Ecc., defined by Aux. Ellipse.
e'	e'		Solar Eccentricity.
		e_0'	Solar Ecc., defined by Solar Aux. Ellipse.
γ†			Tan i, defined by Principal Term in z/a.
γ_1†	γ		Sin $\frac{1}{2}i$, " " " Lat.
		J_0	Inclin., " " " sin U. [For K_0, see Arts. 281, 286.]
		$\bar{f},\ n_0z$	True and Mean Anomalies of Aux. Ell.
		$g,\ n_0\delta z$	Mean and Periodic parts of n_0z.
		s	Sin $U-\sin J_0\sin(\bar{f}+\omega)$.
		$\bar{r}$	Radius vector of Aux. Ellipse.
		$1+\nu$	$r/\bar{r}$.

* Hansen leaves out the zero suffix when the earlier developments have been completed.

† These are not the definitions actually assigned by de Pontécoulant. See Art. 159.

INDEX OF AUTHORS QUOTED.

(*The numbers refer to the pages.*)

GENERAL INDEX.

(*The numbers refer to the pages.*)

The following abbreviations are frequently used:—P. *for* de Pontécoulant's method, D. *for* Delaunay's method, H. *for* Hansen's method, R. *and* rect. coor. method *for* the method of Chapter XI. *These abbreviations refer only to the methods as set forth in the text.*

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